

# A METRIC APPROACH TO ELASTIC REFORMATIONS

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**ABSTRACT.** We study a variational framework to compare shapes, modeled as Radon measures on  $\mathbb{R}^N$ , in order to quantify how they differ from isometric copies. To this purpose we discuss some notions of *weak deformations* termed *reformations* as well as integral functionals having some kind of *isometries* as minimizers. The approach pursued is based on the notion of pointwise Lipschitz constant leading to a space metric framework. In particular, to compare general shapes, we study this *reformation* problem by using the notion of transport plan and of Wasserstein distances as in optimal mass transportation theory.

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## INTRODUCTION

One of the main goal in shape analysis relies in detecting and quantifying differences between shapes. The interest for such studies concerns a wide range of applications, especially those within the computer vision community, in particular in pattern recognition, image segmentation, and face recognition (see [49]). In recent years many authors have focused their attention on the notions of *shape space* and *shape metric*

to the aim of establishing a general framework in which the analysis of shapes crucially depends on their invariance with respect to suitable geometric transformations (see [23, 57]). A natural suggestion in this direction comes from continuum mechanics since the variational theory of elasticity can be used to compare the initial and final shape of a deformable material body, i.e. to establish how the two shapes differ from an isometry of the euclidean space. Therefore some authors begin to study the possible links between elastic energies and distances in shape spaces (see [23, 64, 65]).

On the other side, by arguing from a mechanical perspective, we know that a large class of physical manifestations (fractures, fragmentations, material instabilities) require more general kinematical tools than those available in the context of Sobolev maps, hence it seems reasonable to exploit a more general mathematical framework to obtain more accurate descriptions of more complex physical problems.

In this paper we model (material) shapes as Radon measures on subsets  $X, Y \subset \mathbb{R}^N$  and study a variational model to the aim of quantifying how a target shape  $\nu$  on  $Y$  differs from an isometric copy of  $\mu$  on  $X$ . To this purpose we scrutinize some notions of *weak* deformations, which we denote by the term *reformations*, as well as energy like (or cost) functionals having some kind of *isometries* between  $\mu$  and  $\nu$  as minimizers. In the first part of the paper (Sections 1,2,3) we study the variational problem of *reformation* of two shapes  $\mu$  and  $\nu$  through functions called *reformation maps*, while in the second part (Sections 4,5,6) we relax the problem by considering a formulation in terms of *transport plans* which leads to a variational framework as in optimal transport theory.

For reader convenience we have added an appendix containing some basic tools from analysis in metric spaces.

## DESCRIPTION OF THE VARIATIONAL MODEL AND MAIN RESULTS

The general question addressed in this paper is to provide some variational tools able to *quantify* how two shapes  $X, Y \subset \mathbb{R}^N$  are close to be *isometric*. An usual way to compare the two shapes relies in considering  $Y = u(X)$  for maps  $u$  belonging to a suitable class of admissible maps. The two shapes are *isometric* if there exists  $u : X \rightarrow Y$  such that  $u(X) = Y$  and

$$|u(x) - u(y)| = |x - y|, \quad \forall x, y \in X.$$

Equivalently, the last condition means that the map  $u$  has bi-Lipschitz constant  $L = 1$ . Let us recall that a map  $u : x \rightarrow Y$  is said to be bi-Lipschitz with constant  $L$  if

$$\frac{1}{L}|x - y| \leq |u(x) - u(y)| \leq L|x - y|, \quad \forall x, y \in X.$$

Therefore, the two shapes  $X, Y$  could be considered close to be *isometric* as the bi-Lipschitz constant  $L$  is close to one, so assuming the bi-Lipschitz constant as a quantifier of the closeness to the isometry. This approach has the disadvantage to involve a global condition. For instance, the shapes in Figure 5.2 look very close to be isometric

but the bi-Lipschitz constant is quite large and far from  $L = 1$ , whatever the size of the bending part. To avoid this difficulty some *localization* procedure is needed. This can be done by an analytical approach.

An isometry  $u$  is of course an affine map  $u(x) = Ax + b$  and  $\nabla u = A$  is an orthogonal matrix. Actually, under some regularity assumptions, by Liouville Rigidity Theorem the orthogonality of the Jacobian matrix characterizes the isometric maps (see also Theorem 3.8). Hence, a reasonable way to quantify how two shapes are isometric is that of measuring how  $\nabla u$  is close to be an orthogonal matrix. This program can be carried on by selecting a function  $W$  reaching its minimal value at the orthogonal matrices. Then, by Liouville Rigidity Theorem, it follows that the isometries characterize the minimal possible value of the functional  $I(u) = \int_{\Omega} W(\nabla u) dx$ . This approach is pursued in [65] for smooth 2-dimensional domains where the admissible maps are incompressible diffeomorphisms, i.e.  $u$  such that  $\det(\nabla u) = 1$ .

In order to characterize the isometries, a polyconvex function  $W$  having minimal value at orthogonal matrices is selected. Therefore, to quantify how two domains  $\Omega_1, \Omega_2 \subset \mathbb{R}^N$  are close to be isometric one considers the variational problem

$$\text{minimize } \left\{ \int_{\Omega_1} W(\nabla u) dx \mid u(\Omega_1) = \Omega_2, u \in \mathcal{D} \right\},$$

where  $\mathcal{D}$  denotes the set of incompressible diffeomorphisms. This approach has of course many restrictions. For instance, to compare a connected domain to a disconnected one, or for non-smooth domains, many regularity questions arise.

This discussion is reminiscent of the variational principles in continuum mechanics, indeed in the setting of nonlinear elasticity, in the undeformed state the body occupies a bounded open set  $\Omega \subset \mathbb{R}^N$ , and one usually look for minimizers  $u : \bar{\Omega} \rightarrow \mathbb{R}^N$  of the stored energy  $I(u) = \int_{\Omega} W(\nabla u) dx$  in an admissible class of deformations usually consisting of Sobolev functions which are locally orientation preserving, i.e.  $\det \nabla u(x) > 0$  for a.e.  $x \in \Omega$ .

A main goal of our approach relies in exploiting possible extensions of the variational scheme of elasticity (see also [28] for an analysis of this topic) in order to compare more general shapes as those in Figure 4.1, also allowing *fragmentations*. However, a purely measurable setting does not work to compare shapes as shown in Example 3.6 and, on the other hand, to compare a more extended class of shapes we have to reduce regularity requirements. So, a useful compromise relies in working on a general metric framework.

We remark that an approach like the one followed in [65] cannot be pursued in a metric framework, indeed the mapping  $A \mapsto \varphi(\|A\|)$  is polyconvex only if  $\varphi$  is a positive convex and strictly increasing function (see for instance [15]), therefore the minimal value cannot be reached at orthogonal matrices  $A$ , since they have  $\|A\| = 1$ .

We denote by  $\mathcal{P}(X)$  the space of probability measures on  $X$ . Assume the material shapes are given by probability measures  $\mu \in \mathcal{P}(X)$  and  $\nu \in \mathcal{P}(Y)$ , (to fix ideas consider  $\mu = \mathcal{L}^N \llcorner X$ ,  $\nu = \mathcal{L}^N \llcorner Y$ ).

In this paper we assume the *pointwise Lipschitz constant*  $\text{Lip}(u)(x)$  (see Definition 1.1) as a local descriptor to measure how an admissible map  $u$  is an *expansion* or a *contraction*. Note that the *pointwise Lipschitz constant*  $\text{Lip}(u)(x)$  coincides with the operator norm  $\|\nabla u(x)\|$  whenever  $u$  is a differentiable map. Hence the local expansion and contraction due to the map  $u$  at any point  $x$  are respectively represented by the functions  $e_u(x)$  and  $c_u(x)$  (see Definition 2.1) depending on  $\text{Lip}(u)(x)$ .

We require the admissible maps  $u : X \rightarrow Y$  satisfy the conditions

$$u_{\#}\mu = \nu, \quad (0.1)$$

$$e_u \leq K, \quad c_u \leq H \quad (0.2)$$

for given  $H, K > 0$ . These maps will be called *reformation maps*, and the set of such functions will be denoted by  $\text{Ref}(\mu; \nu)$  (see Definition 2.2). We consider the local reformation energy  $r_u = e_u + c_u$  and the total reformation energy  $\mathcal{R}(u) = \int_X r_u(x) d\mu$ , so in Theorem 3.12 we show that the variational problem

$$\text{minimize}\{\mathcal{R}(u) \mid u \in \text{Ref}(\mu; \nu)\} \quad (0.3)$$

admits solutions whenever  $\text{Ref}(\mu; \nu) \neq \emptyset$ . Therefore, to quantify how the two measures  $\mu, \nu$  are isometric we look to the *elastic reformation energy* between  $\mu$  and  $\nu$  defined by

$$\mathcal{E}(\mu, \nu) := \inf\{\mathcal{R}(u) \mid u \in \text{Ref}(\mu; \nu)\}. \quad (0.4)$$

In Theorem 3.17 we show that the value of (0.4) is attained if and only if the two shapes are isometric.

In Section 4 we extend the scenario to deal with the case of non-existence of maps satisfying (2.4) and this happens, for instance, when fragmentation occurs. In such a case the notion of transport plan coming from the optimal mass transportation or Monge-Kantorovich theory is useful. A transport plan between  $\mu$  and  $\nu$  is a measure  $\gamma \in \mathcal{P}(X \times Y)$ , having  $\mu, \nu$  as marginals, namely  $(\pi_1)_{\#}\gamma = \mu, (\pi_2)_{\#}\gamma = \nu$ , where  $\pi_{1,2}$  are the projections of  $X \times Y$  on its factors. The notion of transport plan could be considered as a generalization of a transport map, i.e.  $u : X \rightarrow Y$ , such that  $u_{\#}\mu = \nu$ , and so as a *weak* notion of reformation of  $\mu$  into  $\nu$ . We shall refer to such measures as *reformation plans*. Actually, to every transport map corresponds the transport plan given by  $(I \times u)_{\#}\mu$ . The shapes  $\mu, \nu$  could be compared by looking the local mass transportation displacement in the target configuration.

More precisely, by Disintegration Theorem (see [3, Section 2.5]) every transport plan  $\gamma \in \mathcal{P}(X \times Y)$  can be written as  $\gamma = f(x) \otimes \mu$ , where

$$f : X \rightarrow (\mathcal{P}(Y), W) \quad (0.5)$$

is called *disintegration map* and  $\mathcal{P}(Y)$  is equipped with the Wasserstein distance  $W$ . This point of view leads to formulate the reformation problem in terms of disintegration map  $f$  and related metric expansion and contraction energies (see Definition 4.6). So, in this setting, reformation maps take value in the space of probabilities, endowed with the Wasserstein metric, over the target domain. The main advantage of this

approach relies in its generality and in its connections with fertile topics as optimal mass transportation and geometric measure theory. However, many interesting open questions arise as the regularity needed on  $f$  to capture relevant geometrical and physical properties of the shapes.

In Section 5 we show several examples of shape reformations attainable by disintegration maps but not attainable by any regular transport map.

In Section 6 we study the main aspects of the variational problems of reformation in the enlarged context of generalized reformations, showing in Theorem 6.4 how isometric measures can be characterized by means of the reformation energy. In Theorem 6.8 we prove the existence of minimizing reformation plans for a constrained variational problem.

## 1. THE POINTWISE LIPSCHITZ CONSTANT

In this section we introduce the notion of *pointwise Lipschitz constant* and scrutiny some properties related to this tool since it will play a crucial role in this paper. Notice that The pointwise Lipschitz constant is a useful tool in theories concerning Sobolev spaces in a metric setting (see [12, 18, 61]).

**Definition 1.1.** Let  $(X, d_X)$ ,  $(Y, d_Y)$  be two metric spaces and let  $f : (X, d_X) \rightarrow (Y, d_Y)$ . The pointwise Lipschitz constant  $\text{Lip}(f)(x_0)$  of  $f$  at  $x_0 \in X$  is defined by

$$\text{Lip}(f)(x_0) := \begin{cases} \limsup_{x \rightarrow x_0} \frac{d_Y(f(x), f(x_0))}{d_X(x, x_0)} & \text{if } x_0 \text{ is a non-isolated point,} \\ 0 & \text{if } x_0 \text{ is an isolated point.} \end{cases} \quad (1.1)$$

It is readily seen that  $\text{Lip}(f)$  is a measurable function.

**Lemma 1.2.** Let  $L > 0$ ,  $X \subset \mathbb{R}^N$  a convex set and let  $f : X \rightarrow (Y, d)$  be a function such that  $\text{Lip}(f)(x) \leq L \forall x \in X$ . Then  $f$  is  $L$ -Lipschitz.

*Proof.* Let  $k > L$  and  $a, b \in X$  be fixed. We define the set

$$S = \{t \in [0, 1] : d(f(a), f(a + t(b - a))) \leq kt|b - a|\}.$$

Since  $\text{Lip}(f)(a) < \mu$  then  $S \neq \emptyset$ . Let  $\tau = \sup S$ . By continuity of  $f$ , which follows by the boundedness of  $\text{Lip}(f)(x)$  (see also [18]), we have that  $\tau \in S$ . We claim that  $\tau = 1$ . Otherwise, since  $\text{Lip}(f)(a + \tau(b - a)) < k$ , we find  $\bar{t} \in (\tau, 1)$  such that

$$d(f(a), f(a + \bar{t}(b - a))) \leq d(f(a), f(a + \tau(b - a))) + d(f(a + \tau(b - a)), f(a + \bar{t}(b - a))) \leq$$

$$\tau k|b - a| + (\bar{t} - \tau)|b - a|k = k\bar{t}|b - a|,$$

so  $\bar{t} \in S$ , in contradiction with the maximality of  $\tau$ . Letting  $k \rightarrow L^+$  we get the thesis.  $\square$

A result similar to the previous lemma holds true for *quasi-convex* metric spaces  $X$  (see [18]).

The pointwise Lipschitz constant is also related to the notion of metric differential (see [7, 42, 43, 50]). A function  $f : X \subset \mathbb{R}^N \rightarrow (Y, d)$  is said to be metrically differentiable at a point  $x_0 \in X$  if there exists a (unique) seminorm on  $\mathbb{R}^N$ , denoted by  $MD(f, x_0)$ , such that for every  $y, z \in X$  the following formula holds true

$$d(f(y), f(z)) - MD(f, x_0)(y - z) = o(\|y - x_0\| + \|z - x_0\|). \quad (1.2)$$

Let  $U \subset \mathbb{R}^N$  be an open set and let  $f : U \rightarrow (Y, d)$  be a Lipschitz function. Hence, for every fixed  $p \in Y$  the function

$$x \mapsto d(f(x), p) : U \rightarrow \mathbb{R}_+ \quad (1.3)$$

is a Lipschitz function and by Rademacher Theorem it is a.e. differentiable in  $U$ . Moreover (see [42, 43]), it turns out that  $f$  is *metrically* differentiable at almost every point.

The following lemma establishes a link between the pointwise Lipschitz constant, the distance function (see [6] for dual Banach spaces) and the metric differential (see (1.7)).

**Lemma 1.3.** *Let  $f : U \subset \mathbb{R}^N \rightarrow (Y, d)$  be a Lipschitz function over a separable metric space  $Y$ . Then, for a.e.  $x_0 \in U$  it results*

$$\text{Lip}(f)(x_0) = \sup_{p \in Y} \|\nabla d(f(x_0), p)\|. \quad (1.4)$$

*Proof.* In the above formula we assume that  $\|\nabla d(f(x_0), p)\| = 0$  if the function  $x \mapsto d(f(x), p)$  is not differentiable at  $x_0$ . For  $p \in Y$  we compute

$$\begin{aligned} \langle \nabla d(f(x_0), p), v \rangle &= \lim_{t \rightarrow 0^+} \frac{d(f(x_0 + tv), p) - d(f(x_0), p)}{t} \leq \\ &\leq \lim_{t \rightarrow 0^+} \frac{d(f(x_0 + tv), f(x_0))}{t} \leq \text{Lip}(f)(x_0). \end{aligned}$$

Taking the supremum with respect to  $v$  and then respect to  $p$ , the inequality

$$\sup_{p \in Y} \|\nabla d(f(x_0), p)\| \leq \text{Lip}(f)(x_0)$$

follows. To get the opposite inequality, we use a slight modification of the proof of [7, Theorem 4.1.6]. Since  $Y$  is separable, we fix a countable dense set  $\{p_n\} \subset Y$ , then for every  $x_1, x_2 \in U$  we have

$$d(f(x_1), f(x_2)) = \sup_n |d(f(x_1), p_n) - d(f(x_2), p_n)|. \quad (1.5)$$

Consider the Lipschitz function  $\varphi_n(t) = d(f(x_0 + tv), p_n)$  and set  $m(t) = \sup_n |\dot{\varphi}_n(t)|$ . Observe that  $|m(t)| \leq \text{Lip}(f)$ . By the Lipschitz condition, we may suppose that, for

every  $n \in \mathbb{N}$ ,  $t = 0$  is a differentiability point for  $\varphi_n$  and also  $t = 0$  is a Lebesgue point for  $m \in L^\infty$ . By (1.5) we obtain

$$\frac{d(f(x_0 + tv), f(x_0))}{t} \leq \sup_n \frac{1}{t} \int_0^t |\dot{\varphi}_n(s)| ds \leq \frac{1}{t} \int_0^t m(s) ds.$$

Letting  $t \rightarrow 0^+$  we get (see Prop. 1 and Th. 2 of [43])

$$MD(f, x_0)(v) \leq m(0) \leq \sup_n \|\nabla d(f(x_0), p_n)\|. \quad (1.6)$$

On the other hand, by (1.2) we get

$$\frac{d(f(x), f(x_0))}{\|x - x_0\|} = MD(f, x_0) \left( \frac{x - x_0}{\|x - x_0\|} \right) + \frac{o(\|x - x_0\|)}{\|x - x_0\|}$$

Letting  $x \rightarrow x_0$ , by (1.6) we get

$$\text{Lip}(f)(x_0) \leq \sup_n \|\nabla d(f(x_0), p_n)\|.$$

□

Observe that by the proof of the previous lemma, the following equality also holds true

$$\text{Lip}(f)(x_0) = \sup_{v \in \mathbb{R}^N, |v|=1} |MD(f, x_0)(v)|. \quad (1.7)$$

**Lemma 1.4.** *Let  $(Y, d)$  be a separable metric space. Assume  $U \subset \mathbb{R}^N$  is an open set,  $(f_n)_{n \in \mathbb{N}}$  be a sequence of (locally) equi-Lipschitz functions  $f_n : U \rightarrow (Y, d)$  and let  $f : U \rightarrow (Y, d)$ . If  $f_n \rightarrow f$  (locally) uniformly on  $U$  then*

$$\int_U \text{Lip}(f)(x) \, dx \leq \liminf_{n \rightarrow +\infty} \int_U \text{Lip}(f_n)(x) \, dx. \quad (1.8)$$

*Proof.* By uniform convergence  $f$  is a (locally) Lipschitz function, moreover we have that  $d(f_n(\cdot), p) \rightarrow d(f(\cdot), p)$  weakly\* in  $W_{loc}^{1,\infty}(U)$ . Therefore, for every  $p$ , by weak l.s.c. of the gradient norm (see also Ch. III Th. 3.3 of [59]) and using (1.4) we get

$$\int_U \|\nabla d(f(x), p)\| dx \leq \liminf_{n \rightarrow +\infty} \int_U \|\nabla d(f_n(x), p)\| dx \leq \liminf_{n \rightarrow +\infty} \int_U \text{Lip}(f_n)(x) \, dx.$$

Since  $Y$  is separable, as in the proof of Lemma 1.3, denoting by  $g_n = \|\nabla d(f(x), p_n)\|$ , by (1.4) we may assume that  $\text{Lip}(f)(x) = \lim_n g_n(x)$ . Moreover, observe that  $|g_n(x)| \leq \text{Lip}(f)$ . Hence, passing to the limit under the integral sign we finally obtain

$$\int_U \text{Lip}(f)(x) \, dx = \lim_{n \rightarrow +\infty} \int_U g_n(x) dx \leq \liminf_{n \rightarrow +\infty} \int_U \text{Lip}(f_n)(x) \, dx.$$

□

*Remark 1.5.* The above Lemma holds true of course for the function  $\text{Lip}^p(f)$ , for any  $p \geq 1$ . If  $Y \subset \mathbb{R}^N$ , the uniform convergence can be replaced by the weak convergence on the Sobolev space  $W^{1,p}(U)$ . In such a case, Lemma 1.4 just states the lower semicontinuity of the  $p$ -Dirichlet energy in Sobolev spaces, since if  $u : \mathbb{R}^N \rightarrow \mathbb{R}^N$  is differentiable at  $x$  then  $\text{Lip}(u)(x) = \|\nabla u(x)\|$ , see also [59, Ch. 3 Theorem 3.3]. For a related semicontinuity result see also [60]. Notice that if  $U \subset \mathbb{R}^N$  is replaced by a general metric space  $(X, d_X)$  the above arguments cannot be applied. In this case of generality we are forced to deal with metric Sobolev spaces (see Appendix B).

## 2. REFORMATION MAPS

In this section we introduce the class of *reformation maps* and establish some properties of these functions. Though the definition of reformation map holds for general metric measure spaces, as a first step we restrict our analysis to the euclidean framework of subsets of  $\mathbb{R}^N$ .

To fix ideas consider  $\Omega \subset \mathbb{R}^N$  an open bounded connected set,  $X = \overline{\Omega}$ ,  $Y \subset \mathbb{R}^N$ . Moreover,  $\mu, \nu$  are given Radon measures on  $X$  and  $Y$  respectively, with normalized masses, so that  $\mu \in \mathcal{P}(X)$  and  $\nu \in \mathcal{P}(Y)$ .

**Definition 2.1** (Expansion and contraction energy). *Let  $x_0 \in X$  and  $u : X \rightarrow Y$ . The pointwise expansion energy of  $u$  at  $x_0$  is defined by*

$$e_u(x_0) := \text{Lip}(u)(x_0) = \limsup_{x \rightarrow x_0} \frac{|u(x) - u(x_0)|}{|x - x_0|}. \quad (2.1)$$

*The pointwise contraction energy of  $u$  at  $x_0$  is defined by*

$$c_u(x_0) := \limsup_{x \rightarrow x_0} \frac{|x - x_0|}{|u(x) - u(x_0)|}. \quad (2.2)$$

*The pointwise reformation energy of  $u$  at  $x_0$  is defined by*

$$r_u(x_0) = e_u(x_0) + c_u(x_0). \quad (2.3)$$

**Definition 2.2** (Reformation maps). *We shall call reformation map any map  $u : X \rightarrow Y$  such that the following conditions hold true:*

$$u_{\#}\mu = \nu, \quad (2.4)$$

$$\forall x \in X \ \exists H, K, r > 0 \text{ s.t. } e_u(y) \leq K, \ c_u(y) \leq H \ \forall y \in \overline{B}(x, r) \cap X. \quad (2.5)$$

*We shall denote by  $\text{Ref}(\mu; \nu)$  the set of reformation maps between  $\mu$  and  $\nu$ .*

The point in the above definition is that the constants  $H, K$  may depend just on the point  $x$  and not by the map  $u \in \text{Ref}(\mu; \nu)$  or the radius  $r > 0$ . Therefore, the reformation maps are characterized by locally bounded expansion and contraction. Notice that, by the bounds (2.5), any  $u \in \text{Ref}(\mu; \nu)$  is continuous and, by Stepanov Theorem (see [34]), is a.e. differentiable in  $\Omega$ . In particular, by Lemma 1.2 reformation maps are locally Lipschitz on  $\Omega$ .



*Remark 2.3.* In a mechanical perspective, the constraints stated in (2.5) could be considered as a bound on the maximum expansion or contraction experienced by the material  $\Omega$ . In this setting, the assumption that of the constants  $H, K$  do not depend on the map  $u$  in (2.5) corresponds to a constitutive property of the material under consideration. We point out that some bounds as in (2.5) are in some sense necessary to control the geometry of the reformations. For instance, in the case of  $\nu = \delta_{y_0}$  we have  $e_u = 0$ ,  $c_u = +\infty$  for any map  $u$  satisfying (2.4). On the other hand, mapping a bar into a bended one (see Fig. 2.1) by two piecewise isometries  $u_1, u_2$  such that  $u_1(x_0) \neq u_2(x_0)$ , we necessarily create a *fracture* at the point  $x_0$ . It results  $e_u(x_0) = +\infty$  at the discontinuity point  $x_0$ . See also Example 3.6. Therefore, roughly speaking, the bound  $c_u \leq H$  means no collapsing, while  $e_u \leq K$  means no fractures.

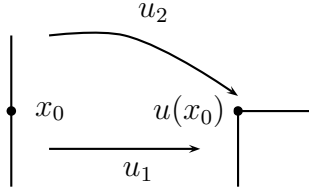


FIGURE 2.1. Mapping a bar into a bended one.

*Remark 2.4.* The constraint  $c_u \leq H$  in (2.5) is related to inversion properties, both local or global, of reformation maps, see [24, 25, 40]. Observe that for differentiable maps with non-vanishing Jacobian we always have

$$c_u = \|(\nabla u)^{-1}\|, \quad e_u = \|\nabla u\|.$$

Therefore, it would be an interesting question to consider pointwise constants  $H, K$  depending also on the map  $u$ . This is similar in spirit to the passage from functions with bounded distortion to functions with finite distortion (see the monograph [39]). However, in such a case, inversion properties becomes more subtle and further assumptions are needed, see for instance [46, 47, 58] for inversion results of Sobolev maps. Anyway, the main interest of reformation maps relies in this perspective in considering just metric objects. See Section 4 for an extension in a metric setting. However, in a purely metric framework, such pointwise conditions are not enough to guarantee inversion properties. Consider for instance the map  $u : \mathbb{R} \rightarrow \mathbb{R}, u(x) = |x|$  having  $e_u = c_u = 1$  at every point. Therefore, also uniform bounds alone are not enough to get satisfactory inversion properties. Under differentiability assumptions in  $\mathbb{R}^N$ , pointwise bounds are in fact enough, see Lemma 2.9. However, in general this is not true. For instance, see [54], it is possible to find everywhere differentiable maps with everywhere invertible differential on Hilbert spaces which are not neither open

or locally one-to-one. In the metric setting different restrictions arise, see [25] for a detailed discussion.

*Remark 2.5.* The uniform bounds (2.5) are also related to quasi-isometries, see for instance [9, 40]. In such case uniform bounds

$$m \leq D^- f(x) \leq D^+ f(x) \leq M,$$

where

$$D^- f(x_0) = \liminf_{x \rightarrow x_0} \frac{|f(x) - f(x_0)|}{|x - x_0|}, \quad D^+ f(x_0) = \limsup_{x \rightarrow x_0} \frac{|f(x) - f(x_0)|}{|x - x_0|}$$

denotes the local distortion of distances, are required. Observe that  $e_u(x) = D^+ f(x)$ , while  $c_u(x) = \frac{1}{D^- f(x_0)}$ . Indeed,

$$\frac{|x - x_0|}{|f(x) - f(x_0)|} = \frac{1}{\frac{|f(x) - f(x_0)|}{|x - x_0|}} \leq \frac{1}{\inf_B \frac{|f(x) - f(x_0)|}{|x - x_0|}}.$$

Taking the supremum over  $B = B(x_0, r)$  and then letting  $r \rightarrow 0^+$  we get  $c_u(x) \leq \frac{1}{D^- f(x_0)}$ . In a similar way the opposite inequality follows.

*Remark 2.6.* The mass conservation property (2.4) is a generalized version of incompressibility and it can be always satisfied (provided  $\mu$  has no atom, see [53]) by some measurable map  $u$ . Actually, condition (2.4) is equivalent to the following change of variable formula

$$\int_X f(u(x)) \, d\mu = \int_Y f(y) \, d\nu, \quad (2.6)$$

for every continuous function  $f : Y \rightarrow \mathbb{R}$ .

Since  $\text{Ref}(\mu; \nu)$  is made by *nice* functions, to compare the present approach with the classical ones (see also [65]), one should also require the condition  $u(X) = Y$  in the case of  $\mu = \mathcal{L}^N \llcorner X$ ,  $\nu = \mathcal{L}^N \llcorner Y$ . However, we point out that this surjection requirement is actually a severe constraint. Indeed, for instance, to find Lipschitz functions  $u : X \rightarrow Y$ ,  $u(X) = Y$  for general compact sets in dimension  $N \geq 3$  (see [1]), as far as we know, is still an open question. Moreover, also bi-Lipschitz functions between *nice* sets are not easy to find (see [20, 31]). Anyway, in such a case many restrictions on the target space  $Y$  may be needed (connectedness, for instance).

In the following we prove some properties enjoyed by reformation maps. A first estimate easy to verify is an immediate consequence of Definition 2.2 and is given by the following proposition.

**Proposition 2.7.** *Let  $u \in \text{Ref}(\mu; \nu)$ . Then, for every  $x_0 \in X$  there exists  $r > 0$  such that*

$$\frac{1}{H} |x - x_0| \leq |u(x) - u(x_0)| \leq K |x - x_0| \quad \forall x \in X \cap \overline{B}(x_0, r). \quad (2.7)$$

**Lemma 2.8.** *Let  $u \in \text{Ref}(\mu; \nu)$ . Then  $u$  is a discrete map, i.e. for every  $y \in Y$ ,  $u^{-1}(y)$  is a finite set.*

*Proof.* Let  $y \in Y$ . By (2.7), if  $x_0 \in u^{-1}(y)$  we have that  $u(x) \neq u(x_0)$  for every  $x \in \overline{B}(x_0, r)$ . Hence,  $x_0$  is an isolated point of  $u^{-1}(y)$ . We claim that  $u^{-1}(y)$  is a finite set. Indeed, otherwise, since  $X$  is compact, we find a sequence  $x_n \rightarrow x_0 \in X$  such that  $x_n \in u^{-1}(y)$ . By continuity of  $u$  we have also  $x_0 \in u^{-1}(y)$ . This is a contradiction since  $x_0$  is an isolated point of  $u^{-1}(y)$ .  $\square$

For discrete continuous maps the *local degree* or *local index*  $i(x_0, u)$  of  $u : X \rightarrow Y$  at  $x_0 \in X$  is defined as follows

$$i(x_0, u) = \deg(u(x_0), u, \overline{B}(x_0, r)), \quad (2.8)$$

where  $\deg(y, u, B)$  denotes the topological degree (we refer to [19, 59] for an introduction to degree theory).

Let  $u^{-1}(y) = \{x_1, \dots, x_h\}$ , we have the following relation

$$\deg(y, u, Y) = \sum_{j=1}^h i(x_j, u). \quad (2.9)$$

Observe that if  $u$  is locally injective in a neighborhood of  $x \in X$  then  $|i(x, u)| = 1$ . We say that  $u$  is a *sense-preserving (reversing) continuous map* if the local index  $i(x, u)$  has constant sign in  $X$ . Notice that each homeomorphism on a domain is either sense-preserving or sense-reversing (see [19, Theorem 3.35]). Moreover, sense-preserving or sense-reversing differentiable maps have constant Jacobian sign, (see [19, Lemma 5.9]), since

$$i(x_0, u) = \text{sign}(Ju(x_0)), \quad (2.10)$$

where  $Ju := \det \nabla u$ , providing  $Ju(x_0) \neq 0$ .

**Lemma 2.9.** *Let  $u \in \text{Ref}(\mu; \nu)$ . If  $u$  is differentiable at  $x_0 \in \Omega$  then  $Ju(x_0) \neq 0$ .*

*Proof.* Suppose by contradiction that  $Ju(x_0) = 0$ . Then we find a vector  $|v| = 1$  such that  $\nabla u(x_0) \cdot v = 0$ . Fixed  $\varepsilon > 0$ , since  $u$  is differentiable, there exists  $\delta > 0$  such that  $|u(x_0 + tv) - u(x_0)| < \varepsilon t$ , whenever  $|t| < \delta$ . On the other hand, by (2.7), there exists  $0 < t < \delta$  such that  $\frac{t}{H} < |u(x_0 + tv) - u(x_0)| < \varepsilon t$ . Hence  $\frac{1}{H} < \varepsilon$ . Letting  $\varepsilon \rightarrow 0^+$  we get a contradiction.  $\square$

By the previous lemma, any  $u \in C^1(X; Y) \cap \text{Ref}(\mu; \nu)$  (or even an everywhere differentiable reformation map) is locally invertible on  $\Omega$  (see [32] for a related inversion result and [54] for an elementary analytical proof).

If  $u$  is only a.e. differentiable, by Lemma 2.9 we have  $Ju(x) \neq 0$  for a.e.  $x$ . However, it is well known that in general this condition does not ensure the local invertibility of Sobolev maps (see for instance [46]). By the way, the condition  $Ju > 0$  on an open set  $\Omega$  is a standard requirement (see for instance [58]), ensuring that  $u$  is locally

invertible for a.e.  $x \in \Omega$ . The restriction to sense-preserving maps is also made in [25] to derive local inversion problem. To this aim, also assumptions on the boundedness of  $HK \leq M$ , for sufficiently small  $M$  are necessary. In our context, since we are interested in comparing domains, also in a metric framework, restrictions to open maps seem more natural.

For reader convenience we recall the following well known result

**Theorem 2.10.** *Let  $u : \Omega \rightarrow \mathbb{R}^N$  be a continuous open and discrete map. Then  $u$  is sense-preserving or sense-reversing.*

*Proof.* We essentially follow [8]. First, by Cernavskii Theorem (see also [32]) it follows that  $\dim(u(B_u)) \leq N - 2$  and  $\dim(B_u) \leq N - 2$ , where  $B_u$  denotes the branch set of  $u$ , i.e. the set of points  $x \in \Omega$  such that  $u$  is not a local homeomorphism at  $x$ , while the dimension  $\dim$  is understood in the sense of [37]. Here, of course, we assume  $N \geq 2$ . For  $N = 1$  the result is not true as it happens for instance for  $u(x) = |x|$ . Since  $\dim(\Omega) = N$ , it follows (see [37, Theorem IV]) that  $\Omega \setminus B_u$  is connected. Since  $u$  is a local homeomorphism on  $\Omega \setminus B_u$  we have

$$i(x, u) = c \quad \forall x \in \Omega \setminus B_u, \quad (2.11)$$

where either  $c = 1$  or  $c = -1$ . Let  $x_0 \in B_u$  and  $y_0 = u(x_0)$ . By (2.7) we find a closed ball  $C := \overline{B}(x_0, r)$  such that  $u(x) \neq u(x_0) \quad \forall x \in \overline{B}(x_0, r) \setminus \{x_0\}$ . Also, in a neighborhood  $V$  of  $y_0$  we have

$$\deg(y, u, C) = i(x_0, u) \quad \forall y \in \overline{V}.$$

Consider now  $U = u^{-1}(V)$  which is open in  $\Omega$ . Since  $u$  is open, it follows that  $u(U)$  is also open in  $u(\Omega)$ . Since  $\dim(u(B_u)) \leq N - 2$  we easily find  $y^* \in u(U) \setminus u(B_u)$ . Let  $u^{-1}(y^*) \cap C = \{x_1, \dots, x_k\} \subset \Omega \setminus B_u$ . Therefore, by (2.11)  $i(x_i, u) = c$ , for every  $i = 1, \dots, k$  and, since  $y^* \in V$ , by (2.9) we compute

$$i(x_0, u) = \deg(y^*, u, C) = \sum_{i=1}^k i(x_i, u) = kc.$$

The above formula shows that the index  $i(x, u)$  has constant sign.  $\square$

*Remark 2.11.* The previous theorem holds true in the case  $X, Y$  are topological manifolds.

The following statement employs the same arguments of [59, Ch. II Theorem 6.6].

**Lemma 2.12.** *Let  $u : \Omega \rightarrow \mathbb{R}^N$  be a discrete sense-preserving (reversing) continuous map such that  $|i(x_0, u)| = 1$ . Then  $u$  is injective in a neighborhood of  $x_0$ .*

*Proof.* Suppose  $i(x_0, u) = 1$ . The other case is analogous. By contradiction, if  $u$  is not injective we have two distinct sequences  $(x_n^1)_{n \in \mathbb{N}}, (x_n^2)_{n \in \mathbb{N}}$  converging to  $x_0$  such that for every  $n \in \mathbb{N}$ :  $u(x_n^1) = u(x_n^2) = y_n$ . By continuity we also have  $u(x_n^1) \rightarrow y_0 = u(x_0)$ .

Since the degree is constant in a neighborhood of  $y_0$ , for  $n \in \mathbb{N}$  large enough and suitable small radius  $r > 0$  we have

$$\deg(y_n, u, \overline{B}(x_0, r)) = \deg(y_0, u, \overline{B}(x_0, r)) = i(x_0, u) = 1. \quad (2.12)$$

On the other hand, since  $u$  is sense-preserving and by (2.9) we have

$$\deg(y_n, u, \overline{B}(x_0, r)) \geq i(x_n^1, u) + i(x_n^2, u) = 2,$$

contradicting (2.12).  $\square$

We get the following inversion properties.

**Theorem 2.13** (Incompressible maps are invertible). *Assume  $u \in \text{Ref}(\mu; \nu)$  is an incompressible map (see for instance [65]), i.e.*

$$|Ju(x)| = 1 \text{ for a.e. } x \in \Omega. \quad (2.13)$$

*Then  $u|_\Omega$  is globally invertible.*

*Proof.* Of course, condition (2.4) holds true for injective such maps  $u$ . Anyway, let us begin by showing that  $u$  is an open map. Let  $U$  be an open subset of  $\Omega$ . We have to prove that  $V = u(U)$  is open. Let  $y_0 \in V$ ,  $y_0 = f(x_0)$  for a  $x_0 \in U$ . By (2.7) we find  $C = \overline{B}(x_0, r)$  such that  $y_0 \notin u(\partial C)$ . Therefore, it is well defined  $\deg(y_0, u, C)$  which is constant in a neighborhood of  $y_0$ . If  $u$  is differentiable at  $x_0$ , by Lemma 2.8 and Lemma 2.9 we have

$$|\deg(y, u, C)| = |\deg(y_0, u, C)| = 1,$$

in a neighborhood  $V_{y_0}$  of  $y_0$ . Since  $\deg(y, u, C) \neq 0$ , it follows that  $V_{y_0} \subset u(C) \subset V$ , since otherwise the degree would be zero.

Observe that since  $\chi_B(x) \leq \chi_{u(B)}(u(x))$ , by Lusin Theorem and (2.6), we have

$$\mathcal{L}^N(B) = \int_X \chi_B(x) dx \leq \int_X \chi_{u(B)}(u(x)) dx = \int_Y \chi_{u(B)} u(y) dy = \mathcal{L}^N(u(B)). \quad (2.14)$$

On the other hand, denoting by  $N(y, u, K) := \text{card}\{u^{-1}(y) \cap K\}$  the multiplicity function, by the Area Formula we compute

$$\mathcal{L}^N(u(B(x_0, r))) \leq \int_{\mathbb{R}^N} N(y, u, B(x_0, r)) dy = \int_{B(x_0, r)} |Ju| dy = \mathcal{L}^N(B(x_0, r)).$$

Therefore, for a.e.  $y \in u(B(x_0, r))$ , it results  $N(y, u, B(x_0, r)) = 1$ . Hence, if  $x_0$  is a non-differentiability point, by the Lusin (N)-property, there exists a differentiability point  $x \in B(x_0, \delta)$  of  $u$  and an open neighborhood  $V_{y_0}$  of  $y_0$  such that  $u(x) \in V_{y_0}$  and  $\deg(u(x), u, C) \neq 0$ . As above it follows that  $V_{y_0} \subset u(C) \subset V$ . Hence  $u|_\Omega$  is open. By Theorem 2.10 we have that  $u$  is sense-preserving, or sense-reversing. Since actually it results  $|i(x_0, u)| = 1$ , the statement follows by Lemma 2.12.  $\square$

*Remark 2.14.* For reformation maps satisfying (2.13), it also results  $|i(x, u)| = 1$  for a.e.  $x \in \Omega$ . Therefore, by Lemma 2.12 such maps  $u$  are a.e. locally invertible on  $\Omega$ . As previously discussed, actually to obtain this invertibility property it is enough to require  $|Ju| \leq 1$  a.e. in  $\Omega$ . Since  $|Ju| \leq \|\nabla u\|^N$ , this happens for instance for reformation maps  $u$  satisfying the condition  $e_u \leq 1$ .

**Theorem 2.15** (Small reformations are invertible). *Let  $u \in \text{Ref}(\mu; \nu)$  be such that  $e_u < \sqrt[N]{2}$ . Then  $u$  is globally invertible.*

*Proof.* By the constraint  $c_u \leq H$  we find (see [51, Prop. 1.1, Sec. 3]) an open dense subset  $U \subset X$  on which  $u$  is locally bi-Lipschitz. It follows that the multiplicity function  $N(y, u, U)$  is locally constant (see [2]). We first prove that  $u$  is globally invertible on  $U$ . For  $y = u(x) \in u(U)$ , we have to prove that  $N(y, u, U) = 1$ . Observe that by the Domain Invariance Theorem,  $u|_U$  is open. Let  $B = B(x, r)$  be a ball on which  $u$  is bi-Lipschitz. We may suppose that  $D = N(y, u, B)$  is constant on  $u(B)$ . Since  $|Ju| \leq \|\nabla u\|^N$ , by the Area formula we compute

$$D\mathcal{L}^N(u(B)) = \int_{\mathbb{R}^N} N(y, u, B) dy = \int_B |Ju(x)| dx \leq \int_B \|\nabla u\|^N dx = \int_B e_u(x)^N dx. \quad (2.15)$$

Observe that by the push-forward condition, as in (2.14), we obtain  $\mathcal{L}^N(B) \leq \mathcal{L}^N(u(B))$ . Therefore, if the map  $u$  satisfies the following small expansion condition

$$\int_B e_u(x)^N dx < 2\mathcal{L}^N(B), \quad (2.16)$$

then by (2.16) and (2.15) we get

$$D\mathcal{L}^N(u(B)) < 2\mathcal{L}^N(B) \leq 2\mathcal{L}^N(u(B)),$$

hence the map  $u$  is globally invertible on  $U$ . By uniform continuity,  $u$  uniquely extends to the whole  $X$  and therefore letting global invertibility on  $X$ .  $\square$

*Remark 2.16.* By Area Formula, for  $B = B(x_0, r)$  using (2.15) it results

$$\int_B e_u(x)^N dx \geq \frac{\mathcal{L}^N(u(B))}{\mathcal{L}^N(B)}.$$

By Lebesgue Differentiation Theorem this yields the a.e. inequality

$$e_u(x_0) \geq \left( \lim_{r \rightarrow 0^+} \frac{\mathcal{L}^N(u(B))}{\mathcal{L}^N(B)} \right)^{\frac{1}{N}} := \mu_u(x_0)^{\frac{1}{N}}.$$

The volume derivative  $\mu_u(x_0)$  is also related to *quasiconformal maps*. We refer to [34, 44], see also [27].

**Theorem 2.17.** [25, Th. II] *Let  $u \in \text{Ref}(\mu; \nu)$  be an open map such that  $HK < 2$ . Then  $u|_\Omega$  is locally invertible.*

*Proof.* By Lemma 2.8 and Theorem 2.10 it follows that  $u$  is sense-preserving or sense-reversing. Hence, [25, Th. II] applies.  $\square$

The condition  $e_u c_u \leq M$  may be required to hold just a.e. by reducing the upper bound  $M < \sqrt[N]{2}$  (see [25]). Observe that the results of [25] hold for dimension  $N \geq 2$ . Anyway, the map  $u(x) = |x|$  in one dimension is not a counterexample to Theorem 2.15 since  $u$  is not mass preserving around  $x_0 = 0$ . If  $\Omega$  is a ball, or in some classes of convex sets, for sufficiently small  $M$  the map  $u$  is actually globally invertible (see [26, 27]).

### 3. THE VARIATIONAL PROBLEM OF ELASTIC REFORMATION

**Definition 3.1.** Let  $u : X \rightarrow Y$ ,  $\mu \in \mathcal{P}(X)$ ,  $\nu \in \mathcal{P}(Y)$ , such that  $u_{\#}\mu = \nu$ . We define the total reformation energy  $\mathcal{R}(u)$  of a reformation map  $u$  of  $\mu$  into  $\nu$  as follows

$$\mathcal{R}(u) := \int_X r_u(x) \, d\mu. \quad (3.1)$$

To ensure that  $\mathcal{R}(u) < +\infty$  for every  $u \in \text{Ref}(\mu; \nu)$ , we will always assume

$$H(x), K(x) \in L^1(X, \mu), \quad (3.2)$$

where  $H, K$  are given in Definition 2.2. We have the following

**Lemma 3.2.**

$$c_u(x) \geq \frac{1}{e_u(x)} \quad \forall x \in X. \quad (3.3)$$

*Proof.* It suffices to recall that  $c_u(x) = \frac{1}{D-f(x)}$ , see Remark 2.5.  $\square$

Observe that for every  $u : X \rightarrow Y$  such that  $u_{\#}\mu = \nu$ , by Lemma 3.2 it results  $\mathcal{R}(u) \geq 2$ .

Actually, Definition 3.1 is motivated by the trivial fact that the real function  $f(x) = x + 1/x$  reaches its minimum value at  $f(1) = 2$ . Moreover, observing that at any  $x_0 \in X$

$$r_u(x_0) \geq e_u(x_0) + \frac{1}{e_u(x_0)} \geq 2, \quad (3.4)$$

we have that  $r_u(x_0)$  reaches its minimum value if  $u : X \rightarrow Y$  is an isometric mapping, i.e.  $|u(x) - u(y)| = |x - y|$ ,  $\forall x, y \in X$ . Therefore  $\mathcal{R}(u)$  can be viewed as a measure detecting how  $u$  is far from being an isometric map.

A useful property is the following

**Lemma 3.3.** Let  $u : X \rightarrow Y$  be a local homeomorphism. Then

$$c_u(x) = e_{u^{-1}}(u(x)) \quad \forall x \in X. \quad (3.5)$$

*Proof.* Fix  $x \in X$ ,  $\delta_1 > 0$  and let  $B_1 = B(u(x), \delta_1)$ . By the local homeomorphism condition, there exists a  $\delta > 0$  such that  $u(B_\delta) \subset B_1$  and  $u$  is invertible on  $B_\delta = B(x, \delta)$ . For every  $y \in B_\delta$  we have

$$\frac{|y - x|}{|u(y) - u(x)|} = \frac{|u^{-1}(u(y)) - u^{-1}(u(x))|}{|u(y) - u(x)|} \leq \sup_{z \in B_1} \frac{|u^{-1}(z) - u^{-1}(u(x))|}{|z - u(x)|}.$$

Taking the supremum with respect to  $y \in B_\delta$  and letting  $\delta_1 \rightarrow 0^+$ , we get  $c_u(x) \leq e_{u^{-1}}(u(x))$ . Analogously we deduce the opposite inequality.  $\square$

**Definition 3.4.** We define the elastic reformation energy between  $\mu$  and  $\nu$  as

$$\mathcal{E}(\mu, \nu) := \inf\{\mathcal{R}(u) \mid u \in \text{Ref}(\mu; \nu)\}. \quad (3.6)$$

Since the Monge transport problem is not symmetric, in general also the above elastic reformation energy is not symmetric. For instance, transporting a ball, say, into a Dirac delta we have  $\mathcal{E}(\mu, \nu) = +\infty$ . Reversing the shapes, we see that  $\mathcal{E}(\nu, \mu)$  has no meaning simply because  $\text{Ref}(\nu; \mu) = \emptyset$ . Moreover, also in nice cases, the matter is that transport maps could be not invertible. Assuming invertibility for  $u \in \text{Ref}(\mu; \nu)$ , setting  $v := u^{-1}$ , by using Lemma 3.3 we have

$$\begin{aligned} \mathcal{R}(u) &= \int_X e_u \, d\mu + \int_X c_u \, d\mu = \int_X e_u(u^{-1}(u(x))) \, d\mu + \int_X e_{u^{-1}}(u(x)) \, d\mu = \\ &= \int_Y e_{v^{-1}}(v(y)) \, d\nu + \int_Y e_v(y) \, d\nu = \int_Y c_v(y) \, d\nu + \int_Y e_v(y) \, d\nu = \mathcal{R}(v). \end{aligned}$$

Since  $v \in \text{Ref}(\nu; \mu)$ , we get  $\mathcal{E}(\mu, \nu) = \mathcal{E}(\nu, \mu)$ . Therefore, symmetry issues essentially correspond to invertibility of maps.

The question is now to establish conditions in order the infimum in (3.6) is attained. We also want to prevent pathological situations as the one described in Example 3.6 below in which the map  $u : X \rightarrow Y$  is merely a.e. continuous (it is actually a.e. invertible and differentiable).

In order to select isometric maps through a variational property, we need to characterize the maps  $u : X \rightarrow Y$  such that  $r_u(x_0) = 2$ .

**Lemma 3.5.** Let  $x_0 \in X$ ,  $u : X \rightarrow Y$ . Then  $r_u(x_0) = 2$  if and only if

$$\forall \varepsilon > 0 : \frac{1}{1 + \varepsilon} |x - x_0| \leq |u(x) - u(x_0)| \leq (1 + \varepsilon) |x - x_0|, \quad \forall x \in X \cap B(x_0, r_\varepsilon).$$

*Proof.* Assume  $r_u(x_0) = 2$ , then

$$2 = e_u(x_0) + c_u(x_0) \geq e_u(x_0) + \frac{1}{e_u(x_0)} \geq 2,$$

so

$$e_u(x_0) + \frac{1}{e_u(x_0)} = 2 \Rightarrow (e_u(x_0) - 1)^2 = 0 \Rightarrow e_u(x_0) = c_u(x_0) = 1.$$



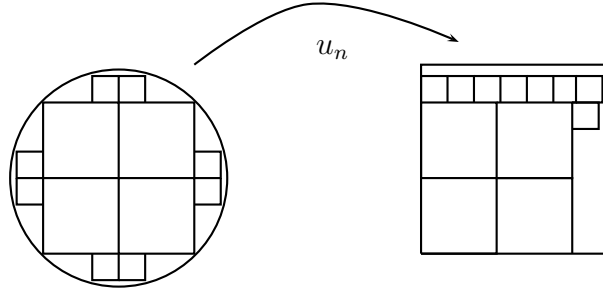


FIGURE 3.1. A piece-wise isometric map for the circle into a square.

Fix  $\varepsilon > 0$ , then  $e_u(x_0) < 1 + \varepsilon$  implies that  $u$  satisfies

$$|u(x) - u(x_0)| \leq (1 + \varepsilon)|x - x_0|, \quad \forall x \in X \cap B(x_0, r_\varepsilon). \quad (3.7)$$

By using the condition  $c_u(x_0) < 1 + \varepsilon$ , eventually by decreasing the radius  $r_\varepsilon$ , we get the opposite inequality. Vice versa, if both the inequalities locally hold, then it results  $2 \leq r_u(x_0) = e_u(x_0) + c_u(x_0) \leq 1 + 1 = 2$ .  $\square$

Therefore, the maps  $u : X \rightarrow Y$  such that  $r_u = 2$  are in some sense *pointwise* locally quasi-isometric, (see [52] for the relation with quasi-conformal maps).

In the following we shall try to characterize in a more precise way the reformation maps  $u \in \text{Ref}(\mu; \nu)$ , if any, realizing the minimum energy level  $\mathcal{R}(u) = 2$ . It is easily seen that

$$\mathcal{R}(u) = 2 \text{ if and only if } r_u(x) = 2 \quad \text{for } \mu - a.e. x \in X. \quad (3.8)$$

In particular, by (3.4) (see also the proof of Lemma 3.5) it results  $e_u(x) = 1$  for  $\mu$ -a.e.  $x \in X$ . Moreover,  $e_u(x) < +\infty$  implies  $u$  continuous at  $x$ . Then these reformation maps  $u$  are at least a.e. continuous functions. However, this mild regularity is too poor to preserve geometric (or physical) properties as we show in the next example.

*Example 3.6.* Let  $\Omega \subset \mathbb{R}^N$  be a smooth bounded open set and  $Q \subset \mathbb{R}^N$  be a cube such that  $\mathcal{L}^N(\Omega) = \mathcal{L}^N(Q)$ , see figure 3.1. For  $n \geq 1$  large enough,  $\Omega$  contains a certain number of disjoint squares  $Q_n$  of length  $\frac{1}{n}$ . Then consider the map  $u_n$  which moves by an isometry every square  $Q_n$  inside  $Q$  in a disjoint way. On the remainder of  $\Omega$ , consider the contained squares  $Q_m$ ,  $m > n$ , and then the map  $u_m$  which coincides with  $u_n$  on the squares  $Q_n$  and moves by an isometry the squares  $Q_m$  inside  $Q$  in a disjoint way. By this procedure it is then defined a sequence  $(u_n)_{n \in \mathbb{N}}$ . Taking the limit  $u = \lim_{n \rightarrow +\infty} u_n$  we obtain a measurable map  $u : \Omega \rightarrow Q$  such that  $u_{\#}\mu = \nu$ , where  $\mu = \mathcal{L}^N \llcorner \Omega$ ,  $\nu = \mathcal{L}^N \llcorner Q$ , and  $r_u(x_0) = 2$  for a.e.  $x_0 \in \Omega$ . Therefore, every bounded smooth open set can be reformed into a square at minimal energy.

In order to preserve geometric and physical properties of the shapes under consideration, we then need more regularity on the admissible reformations. We have the following

**Lemma 3.7.** *Let  $x_0 \in \Omega$ ,  $u : X \rightarrow Y$ . If  $u$  is differentiable at  $x_0$  then*

$$r_u(x_0) = 2 \Rightarrow \nabla u(x_0) \in O(N).$$

*Proof.* By (3.4) we have  $c_u(x_0) = e_u(x_0) = 1$ . Hence, for every  $v \in \mathbb{R}^N$ , taking  $x = x_0 + \delta v$  we get

$$c_u(x_0) = e_u(x_0) = 1 \Rightarrow \frac{|\nabla u(x_0) \cdot v|}{|v|} = 1 \Rightarrow \nabla u(x_0) \in O(N).$$

□

By Liouville Theorem (see for instance [13]) it follows that every  $u \in C^1(X; Y)$  such that  $\mathcal{R}(u) = 2$  is actually an isometry. There are several generalizations of Liouville Rigidity Theorem, however (see [13, 17, 22]) these results are not directly applicable in our context since they generally require a constant sign for the Jacobian, as the condition  $\nabla u(x) \in SO(N)$  for a.e.  $x \in X$ . For instance, the map  $u(x) = x$  if  $x_1 \geq 0$  and  $u(x) = (-x_1, x_2, \dots, x_N)$  if  $x_1 \leq 0$  belongs to the Sobolev space  $W^{1,2}(\Omega, \mathbb{R}^N)$ ,  $\nabla u(x) \in O(N)$  for a.e.  $x$ , but  $u$  is not an isometry.

Since reformations have to preserve the volume, we have the following result.

**Theorem 3.8** (Rigidity). *Let  $U \subset \mathbb{R}^N$  be an open connected bounded set. Let  $u : \overline{U} \rightarrow \mathbb{R}^N$  be a continuous, locally Lipschitz, open map such that  $\mathcal{L}^N(U) = \mathcal{L}^N(u(U))$  and satisfying the following conditions*

- (i)  $u(\partial U) \subset \partial u(U)$
- (ii)  $u$  is a.e. differentiable and  $\nabla u \in O(N)$  a.e. on  $U$ .

*Then  $u$  is an affine function.*

*Proof.* By (ii) it follows that  $u$  is locally a 1-Lipschitz function (see [14, Proposition 3.4]). By the Area Formula and (ii) we infer

$$\mathcal{L}^N(u(U)) = \mathcal{L}^N(U) = \int_U |Ju| \, dx = \int_{\mathbb{R}^N} N(y, u, U) \, dy.$$

Therefore, it follows that  $N(y, u, U) = 1$  for a.e.  $y \in u(U)$ . Observe that

$$u(U) \subset \mathbb{R}^N \setminus \partial u(U) \subset \mathbb{R}^N \setminus u(\partial U). \quad (3.9)$$

Therefore, for every  $x \in U$  the topological degree  $\deg(u(x), u, U)$  is well defined. Since  $u$  is a.e. differentiable, for a.e.  $x \in U$  it results (see Lemma 5.9 of [19])

$$|\deg(u(x), u, U)| = |\text{sign}(Ju(x))| = 1.$$

On the other hand, since  $u(U)$  is connected, by (3.9),  $u(U)$  is contained in a connected component of  $\mathbb{R}^N \setminus u(\partial U)$ . Therefore, the degree is constant on  $u(U)$ . It follows that

the sign of the Jacobian  $Ju$  is a.e. fixed. The conclusion follows by Liouville Theorem for Sobolev maps (see for instance [13]).  $\square$

For a related rigidity result involving local homeomorphisms see [59]. For quasi-isometries over Banach spaces see [9, Cor. 14.8].

*Remark 3.9.* Condition (i) holds of course for invertible maps  $u$ . If we are dealing with locally invertible maps, since continuous and locally invertible maps are open maps, actually by (i) the equality  $u(\partial U) = \partial u(U)$  holds true. Moreover, if the map  $u : \partial U \rightarrow \partial u(U)$  is injective, then  $u$  is globally invertible (see also [48]). Another classical condition for global invertibility holds for simply connected, or simply connectedly exhausted, target  $\overline{u(U)}$  (see [2, 59]). Moreover, suppose to have a continuous, locally invertible, surjective function  $u : X \rightarrow Y$  such that  $\nabla u(x) \in O(N)$  for a.e.  $x$ . Then,  $D = N(y, u, U)$  is constant (see [2]) and by Area Formula we have

$$D\mathcal{L}^N(Y) = \int_Y N(y, u, U) dy = \int_X |Ju(x)| dx = \mathcal{L}^N(X).$$

Hence, if  $\mathcal{L}^N(X) = \mathcal{L}^N(Y)$ , it follows that  $N(y, u, U) = 1$  and hence  $u$  is globally invertible.

By restricting to differentiable functions, if  $\nu$  is not isometric to  $\mu$  then for any such reformation  $u$  it results  $\mathcal{R}(u) > 2$ . A natural question is if it is possible to get close to the minimum value by such reformations.

We state in advance some stability properties for the inverse maps.

**Lemma 3.10.** *Let  $f_n : X \rightarrow Y$ ,  $f_n^{-1} : Y \rightarrow X$  be sequences of equi-Lipschitz mappings. If  $\|f_n - f\|_\infty \rightarrow 0$  and  $\|f_n^{-1} - g\|_\infty \rightarrow 0$ , then  $g = f^{-1}$ .*

*Proof.*

$$\begin{aligned} |g(f(x)) - x| &\leq |g(f(x)) - f_n^{-1}(f(x))| + |f_n^{-1}(f(x)) - f_n^{-1}(f_n(x))| \leq \\ &\leq \|g - f_n^{-1}\|_\infty + H|f(x) - f_n(x)| \leq \|g - f_n^{-1}\|_\infty + H\|f - f_n\|_\infty, \end{aligned}$$

where  $H$  is a common Lipschitz constant for  $f_n^{-1}$ . Analogously we compute

$$\begin{aligned} |f(g(y)) - y| &\leq |f(g(y)) - f_n(g(y))| + |f_n(g(y)) - f_n(f_n^{-1}(y))| \leq \\ &\leq \|f - f_n\|_\infty + L|g(y) - f_n^{-1}(y)| \leq \|f - f_n\|_\infty + L\|g - f_n^{-1}\|_\infty, \end{aligned}$$

for a common Lipschitz constant  $L$  for  $f_n$ . Passing to the limit as  $n \rightarrow +\infty$  we get the thesis.  $\square$

**Lemma 3.11.** *Let  $f_n : \Omega \rightarrow Y$  be a sequence of locally invertible mappings such that  $f_n \rightarrow f$  locally uniformly on  $X$ . If the sequence  $(f_n^{-1})_{n \in \mathbb{N}}$  is made of locally equi-Lipschitz mappings, then  $f$  is also locally invertible.*

*Proof.* Fixed  $x_0 \in X$ , suppose by contradiction to get two distinct sequences  $(x_h^1)_{h \in \mathbb{N}}$ ,  $(x_h^2)_{h \in \mathbb{N}}$  converging to  $x_0$  such that  $f(x_h^1) = f(x_h^2) \forall h \in \mathbb{N}$ . Let  $\varepsilon > 0$  be fixed. By uniform convergence, we find a large integer  $n$  such that  $|f_n(x) - f(x)| < \varepsilon$  for every  $x \in B(x_0, r)$ . As in [47, Proposition 7], we may assume that for every  $n \in \mathbb{N}$   $f_n$  is invertible in  $B(x_0, r)$ . Indeed, setting  $y_0 = f(x_0)$ ,  $y_n = f_n(x_0)$ , let  $U$  be the connected component of  $x_0$  contained in  $f^{-1}(B(y_0, r))$ . By uniform convergence, we may assume that  $|f_n(x) - f(x)| < \frac{r}{3}$  for every  $x \in \overline{U}$ . Since  $|y_n - y_0| < \frac{r}{3}$ , there exists a connected component  $Q_n$  of  $x_0$  contained in  $f_n^{-1}(B(y_0, \frac{2r}{3}))$ . We claim that  $\overline{Q_n} \subset U$ . Indeed, if  $x_j \rightarrow x$  with  $x_j \in Q_n$  we have

$$|f(x) - y_0| \leq |f(x) - f_n(x)| + |f_n(x) - y_0| \leq \frac{r}{3} + |f_n(x) - y_0| < r.$$

Therefore  $x \in U$ . By local invertibility of  $f_n$ , every path connecting  $y \in B(y_0, \frac{2r}{3})$  to  $y_0$  may be lifted to a path in  $Q_n$  starting at  $x_0$ . Hence  $f_n(Q_n) = B(y_0, \frac{2r}{3})$ . Since  $f_n$  is locally invertible in  $Q_n$ , by Global Inversion Theorem ([2, Theorem 1.8]), it follows that every  $f_n$  maps homeomorphically  $Q_n$  onto  $B(y_0, \frac{2r}{3})$ .

Let  $U_1$  be the connected component of  $x_0$  contained in  $f^{-1}(B(y_0, \frac{r}{3}))$ . For every  $x_1 \in U_1$  we have

$$|f_n(x_1) - y_0| \leq |f_n(x_1) - f(x_1)| + |f(x_1) - y_0| < \frac{2r}{3}.$$

Therefore,  $U_1 \subset f_n^{-1}(B(y_0, \frac{2r}{3}))$  which implies that  $U_1 \subset Q_n$ . Therefore, the restriction of  $f_n$  to  $U_1$  is injective for every  $n \geq 1$ . Now, for a large  $h$  we may assume  $x_h^1, x_h^2 \in B(x_0, r) \subset U_1$ . Then we compute

$$\begin{aligned} |x_h^1 - x_h^2| &= |f_n^{-1}(f_n(x_h^1)) - f_n^{-1}(f_n(x_h^2))| \leq H|f_n(x_h^1) - f_n(x_h^2)| \leq \\ &H(|f_n(x_h^1) - f(x_h^1)| + |f(x_h^1) - f(x_h^2)| + |f(x_h^2) - f_n(x_h^2)|) \leq 2H\varepsilon, \end{aligned}$$

where  $H$  is a common Lipschitz constant for  $f_n^{-1}$ . By the arbitrariness of  $\varepsilon$  we get the contradiction  $x_h^1 = x_h^2$ .  $\square$

Now we prove the following existence result for the variational problem of elastic reformation.

**Theorem 3.12.** *Let  $\mu \in \mathcal{P}(\Omega)$  and  $\nu \in P(Y)$  so that  $\mu = \mathcal{L}^N \llcorner \Omega$ ,  $\nu = \mathcal{L}^N \llcorner Y$ . Then the variational problem*

$$\text{minimize} \{ \mathcal{R}(u) \mid u \in \text{Ref}(\mu; \nu), u \text{ open s.t. } HK < 2 \} \quad (3.10)$$

*admits solutions whenever  $\{u \in \text{Ref}(\mu; \nu), u \text{ open s.t. } HK < 2\} \neq \emptyset$ .*

*Proof.* Since  $\mu = \mathcal{L}^N \llcorner \Omega$ , we may assume that  $X = \Omega$ . Otherwise, we should assume that  $X \cap \overline{B}(x, r)$  is a convex, or a quasiconvex set. Let  $(u_n)_{n \in \mathbb{N}}$  be a minimizing sequence. Given  $x_0 \in \Omega$ , let  $K, H, r > 0$  as provided by Definition 2.2. It follows that the sequence  $(u_n)_{n \in \mathbb{N}}$  is locally equi-Lipschitz on  $\overline{B}(x_0, r)$ , see Lemma 1.2 or [18]. Therefore, the sequence  $u_n$  is pointwise equicontinuous on  $\Omega$ . By the Ascoli-Arzelà

Theorem we extract a subsequence converging, uniformly on compact subsets of  $\Omega$ , to a continuous map  $u$ . For this continuous limit map  $u : \Omega \rightarrow \mathbb{R}^N$  it is easily seen that  $u_{\#}\mu = \nu$ . It remains to prove that actually  $u \in \text{Ref}(\mu; \nu)$ , namely that  $e_u(x) \leq K(x_0), c_u(x) \leq H(x_0)$  for every  $x \in X \cap \overline{B}(x_0, r)$ . For  $X = \Omega$ , by Lemma 1.2 we get the Lipschitz condition

$$|u_n(x_1) - u_n(x_2)| \leq K(x_0)|x_1 - x_2|$$

for every  $x_1, x_2 \in \overline{B}(x_0, r) \subset \Omega$ . Passing to the limit as  $n \rightarrow +\infty$  and then as  $r \rightarrow 0^+$ , we obtain  $e_u(x) \leq K(x_0)$ . Let  $x_1 \in \overline{B}(x_0, r)$  and  $y_1 = u(x_1)$  be fixed. Observe that by Theorem 2.17 the maps  $u_n$  are locally invertible. Therefore, using Lemma 1.2 and Lemma 3.3 the inverse maps  $u_n^{-1}$  are also equi-Lipschitz. As in Lemma 3.11, we find the homeomorphisms  $u_n : Q_n \rightarrow B(y_1, r_1)$ . By Lemma 3.3 we get  $e_{u_n^{-1}}(y) \leq H(x_0)$  for every  $y \in B(y_1, r_1)$ . Observe that  $u$  is open by the Domain Invariance Theorem. Hence  $u(\Omega)$  is actually an open set. By Lemma 1.2 it follows

$$|u_n^{-1}(y) - u_n^{-1}(y_1)| \leq H(x_0)|y - y_1|, \quad \forall y \in B(y_1, r_1).$$

For a common neighborhood  $x_1 \in U_1 \subset Q_n$  we have

$$|x - x_1| \leq H(x_0)|u_n(x) - u_n(x_1)|.$$

Passing to the limit as  $n \rightarrow +\infty$  and then as  $x \rightarrow x_1$  we get  $c_u(x_1) \leq H(x_0)$ . Therefore  $u \in \text{Ref}(\mu; \nu)$ .

Fixed  $\varepsilon > 0$ , we find  $\delta > 0$  such that  $\int_E (H(x) + K(x)) d\mu < \varepsilon$  whenever  $\mathcal{L}^N(E) < \delta$ . By using a Vitali covering, we cover  $\Omega$ , up to a measurable set  $\mathcal{L}^N(E) = \delta > 0$ , by a finite number of disjoint neighborhoods  $U_i$  on which  $u_n \rightarrow u$  uniformly and invertibility holds according to Lemma 3.10 and Lemma 3.11.

Since  $u_{\#}\mu = \nu$  we compute

$$\begin{aligned} \mathcal{R}(u) &\leq \sum_{i=1}^l \left( \int_{U_i} \text{Lip}(u)(x) d\mu + \int_{U_i} \text{Lip}(u^{-1})(u(x)) d\mu \right) + \int_E (H(x) + K(x)) d\mu \\ &\leq \sum_{i=1}^l \left( \int_{U_i} \text{Lip}(u)(x) d\mu + \int_{u(U_i)} \text{Lip}(u^{-1})(y) d\nu \right) + \varepsilon. \end{aligned}$$

By Lemma 1.4 we get

$$\begin{aligned} \mathcal{R}(u) &\leq \sum_{i=1}^l \liminf_{n \rightarrow +\infty} \left( \int_{U_i} \text{Lip}(u_n)(x) d\mu + \int_{u(U_i)} \text{Lip}(u_n^{-1})(y) d\nu \right) + \varepsilon \\ &\leq \liminf_{n \rightarrow +\infty} \sum_{i=1}^l \left( \int_{U_i} \text{Lip}(u_n)(x) d\mu + \int_{u(U_i)} \text{Lip}(u_n^{-1})(y) d\nu \right) + \varepsilon \\ &\leq \liminf_{n \rightarrow +\infty} \left( \int_{\Omega} e_{u_n}(x) d\mu + \int_{\Omega} c_{u_n}(x) d\mu \right) + \varepsilon = \liminf_{n \rightarrow +\infty} \mathcal{R}(u_n) + \varepsilon \end{aligned}$$

Letting  $\varepsilon \rightarrow 0^+$  we get the thesis.  $\square$

*Remark 3.13.* In a similar way, according to Theorem 2.15 and Theorem 2.13 we obtain existence for the variational problem over the set  $\{u \in \text{Ref}(\mu; \nu), e_u < \sqrt[N]{2}\}$  or the set  $\{u \in \text{Ref}(\mu; \nu), u \text{ incompressible}\}$ .

*Remark 3.14.* If the surjection property  $u(X) = Y$  is required for  $u \in \text{Ref}(\mu; \nu)$ , we may argue as follows. To check that  $u$  is onto, let us fix  $y_0 \in Y$ . Observe that  $u_n^{-1}$  are locally equi-Lipschitz. Arguing as in Lemma 3.11 for the sequence  $u_n^{-1}$  we find a common neighborhood  $B(y_0, r)$  such that

$$u_n^{-1} : B(y_0, r) \rightarrow U \subset\subset \Omega$$

are simultaneously homeomorphisms. Therefore, since  $u_n(X) = Y$ , we find a sequence  $x_n \rightarrow x_0 \in U \subset \Omega$  such that  $u_n(x_n) = y_0$ . Then we have

$$\begin{aligned} |u(x_0) - y_0| &\leq |u(x_0) - u(x_n)| + |u(x_n) - u_n(x_n)| + |u_n(x_n) - y_0| \\ &\leq |u(x_0) - u(x_n)| + \|u - u_n\|_\infty \rightarrow 0 \end{aligned}$$

as  $n \rightarrow +\infty$ .

*Remark 3.15.* Observe that the compactness of  $\text{Ref}(\mu; \nu)$  does not involve the energy functional  $\mathcal{R}$ . In the setting of mappings with bounded distortion compare with [59, Ch. II Section 9]. In the case of  $H, K \in L^p(X)$  the above minimization result could be obtained by using Rellich-Kondrakov compactness in Sobolev spaces and the l.s.c of the  $p$ -Dirichlet energy. The point is that the above reasonings rely just on metric objects and can be generalized also in a metric framework by using the theory of Sobolev spaces over metric spaces (see Appendix B).

*Remark 3.16.* For  $X$  compact, considering finite coverings, it turns out that  $H, K \in L^\infty$ . Therefore, in such a case we may consider  $H, K$  as two universal constants. However the proof of Theorem 3.12 works as well for the non-compact case. It would be interesting to develop an analogous theory under weaker requirement on the functions  $H, K$ .

We then characterize isometric measures by the following

**Theorem 3.17.** *Let  $\mu \in \mathcal{P}(\Omega)$  and  $\nu \in \mathcal{P}(Y)$ , so that  $\mu = \mathcal{L}^N \llcorner \Omega$ ,  $\nu = \mathcal{L}^N \llcorner Y$ , for a given bounded set  $Y$ . Then,  $\mathcal{E}(\mu, \nu) = 2$  if and only if there exists an isometry  $u$  such that  $u_\# \mu = \nu$ .*

*Proof.* By Theorem 3.12 we get a minimizer  $u : \Omega \rightarrow \mathbb{R}^N$  which belongs to  $\text{Ref}(\mu; \nu)$ . By Theorem 2.15 and Remark 2.15 it follows that  $u$  is globally invertible. By Lemma 3.7 and Theorem 3.8 it follows that  $u$  is a local isometry, then (see for instance [9, Th. 14.1]),  $u$  is an isometric map.  $\square$

By Theorem 3.17 we have that by reforming a flat configuration  $\mu$  in a corrugated one  $\nu$  it results  $\mathcal{E}(\mu, \nu) > 2$ . This last fact gives an alternative proof of the so-called Grinfeld instability (see [21]), indeed, by the changing of the geometry, any possible reformation must expand or contract some portion of the body.

#### 4. GENERALIZED REFORMATIONS

The notion of reformation introduced in the previous section has some restrictions, indeed it is easy to show examples, like the one in Figure 4.1, in which every reformation has a large cost while allowing fractures of the body it is possible to map the initial measure by using local isometries.

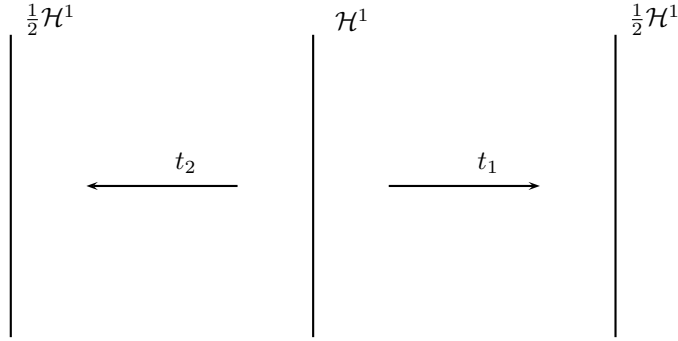


FIGURE 4.1. An isometric fractured reformation.

Here we introduce a notion of *generalized reformation*. Our approach relies on measure theoretic tools mostly developed in the field of optimal mass transportation (see [4, 62, 63]).

The notion of reformation map corresponds to the notion of the so-called transport map, i.e.  $u : X \rightarrow Y$  such that  $u_{\#}\mu = \nu$ . A natural generalization of the transport map is given by the notion of transport plan. A transport plan between two probability measures  $\mu \in \mathcal{P}(X)$  and  $\nu \in \mathcal{P}(Y)$  is a measure  $\gamma \in \mathcal{P}(X \times Y)$  such that  $\pi_{\#}^1 \gamma = \mu$ ,  $\pi_{\#}^2 \gamma = \nu$ , where  $\pi^i$ ,  $i = 1, 2$  denote the projections of  $X \times Y$  on its factors. A transport map  $u$  corresponds to the transport plan  $\gamma_u := (I \times u)_{\#}\mu$ , where  $I$  is the identity map of  $X$ . Observe that the set of transport plans with marginals  $\mu$  and  $\nu$ , denoted by  $\Pi(\mu, \nu)$ , is never empty since it always contains the transport plan  $\mu \otimes \nu$ .

We shall call *generalized reformation*, or *reformation plan*, of  $\mu$  into  $\nu$  any transport plan  $\gamma$  with marginals  $\mu$  and  $\nu$ .

Let us recall some known results which will play a crucial role in the following (we refer to [3, 4]).

**Definition 4.1.** Let  $\mathcal{M}(Y)$  be the space of Radon measures on  $Y$ . A map  $\lambda : X \rightarrow \mathcal{M}(Y)$  is said to be Borel if for any open set  $B \subset Y$  the function  $x \in X \mapsto \lambda_x(B)$  is

a real valued Borel map. Equivalently,  $x \mapsto \lambda_x$  is a Borel map if for any Borel and bounded map  $\varphi : X \times Y \rightarrow \mathbb{R}$  it results that the map

$$x \in X \mapsto \int_Y \varphi(x, y) d\lambda_x$$

is Borel.

**Theorem 4.2** (Disintegration theorem). *Let  $\gamma \in \mathcal{P}(X \times Y)$  be given and let  $\pi^1 : X \times Y \rightarrow X$  be the first projection map of  $X \times Y$ , we set  $\mu = (\pi^1)_\# \gamma$ . Then for  $\mu$ -a.e.  $x \in X$  there exists  $\nu_x \in \mathcal{P}(Y)$  such that*

(i) *the map  $x \mapsto \nu_x$  is Borel,*

$$(ii) \quad \forall \varphi \in \mathcal{C}_b(X \times Y) : \int_{X \times Y} \varphi(x, y) d\gamma = \int_X \left( \int_Y \varphi(x, y) d\nu_x(y) \right) d\mu(x).$$

Moreover the measures  $\nu_x$  are uniquely determined up to a negligible set with respect to  $\mu$ .

Let  $\gamma \in \Pi(\mu, \nu)$ , as usual we will write  $\gamma = \nu_x \otimes \mu$ , assuming that  $\nu_x$  satisfy the condition (i) and (ii) of Theorem 4.2. Obviously the transport plan  $\mu \otimes \nu$  corresponds to the constant map  $x \mapsto \nu_x = \nu$ . Let  $u : X \rightarrow Y$ , observe that for the transport plan  $\gamma_u := (I \times u)_\# \mu$ , the Disintegration Theorem yields  $\gamma_u = \delta_{u(x)} \otimes \mu$ .

*Remark 4.3.* Let  $X \subset \mathbb{R}^N$ , we recall that the barycenter of a measure  $\mu \in \mathcal{P}(X)$  is given by

$$\beta(\mu) = \int_X x d\mu.$$

If  $\gamma = \nu_x \otimes \mu$ , then, by Theorem 4.2 the map  $x \mapsto \beta(\nu_x)$  is measurable. It is possible to define a generalized pointwise expansion and compression energy through the pointwise Lipschitz constant of the map  $x \mapsto \varphi(x) := \beta(\nu_x)$ . Observe that for a transport map  $u$ , since  $\beta(\delta_x) = x$ , we have

$$r_\varphi(x_0) = r_u(x_0).$$

However, it may happen that the map  $\varphi$  is an isometry although the target are quite far from being *isometric* as it happens in Figure 4.2 .

In the sequel we will introduce the notion of generalized pointwise compression and expansion energy through the notion of 1-Wasserstein distance of measures.

**Definition 4.4.** *Let  $\mu, \nu \in \mathcal{P}(X)$ , the 1-Wasserstein distance between  $\mu$  and  $\nu$  is defined by*

$$W(\mu, \nu) = \inf_{\gamma \in \Pi(\mu, \nu)} \int_X d(x, y) d\gamma(x, y). \quad (4.1)$$

Let us recall that by Kantorovich duality (see [4, 29, 62, 63]) the 1-Wasserstein distance between  $\mu$  and  $\nu$  can be expressed as follows

$$W(\mu, \nu) = \sup \left\{ \int_X \varphi d(\mu - \nu) \mid \varphi \in \text{Lip}_1(X) \right\}. \quad (4.2)$$



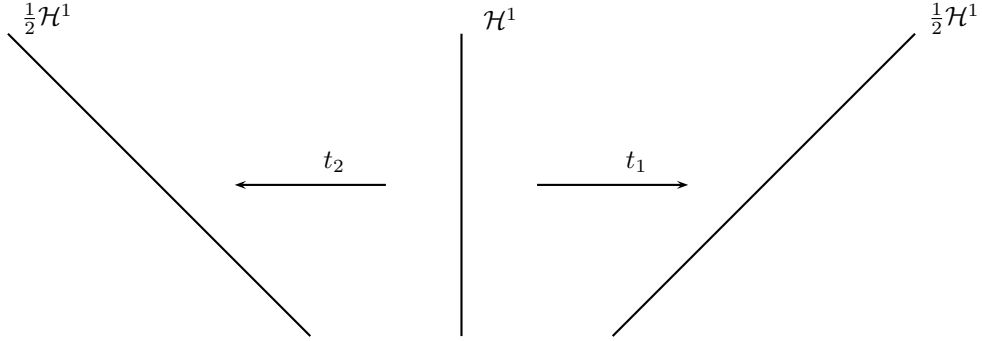


FIGURE 4.2. A barycenter isometric reformation.

**Lemma 4.5.** *The balls of  $(\mathcal{P}(Y), W)$  are convex.*

*Proof.* Let  $\mu \in \mathcal{P}(Y)$ ,  $r > 0$  be fixed, we consider  $\nu_1, \nu_2 \in B := B(\mu, r) \subset \mathcal{P}(Y)$ . For every  $t \in [0, 1]$ , let  $\nu_t := t\nu_1 + (1 - t)\nu_2$ . Then, by considering (4.2), for any fixed  $\varphi \in \text{Lip}_1(Y)$ , we compute

$$\begin{aligned} \int_Y \varphi d(\nu_t - \mu) &= t \int_Y \varphi d(\nu_1 - \mu) + (1 - t) \int_Y \varphi d(\nu_2 - \mu) \\ &\leq tW(\nu_1, \mu) + (1 - t)W(\nu_2, \mu) \leq r \end{aligned}$$

Passing to the supremum with respect to  $\varphi \in \text{Lip}_1(Y)$  we get  $W(\nu_t, \mu) \leq r$ , hence  $\nu_t \in B \forall t \in [0, 1]$ .  $\square$

The above lemma allows to pass from pointwise Lipschitz bounds to local Lipschitz condition over the metric space  $(\mathcal{P}(Y), W)$  (see [18]). Let  $\gamma = \nu_x \otimes \mu$ , the function

$$f : X \rightarrow (\mathcal{P}(Y), W), \quad f(x) = \nu_x. \quad (4.3)$$

will be called *disintegration map*. Let us introduce the notion of generalized compression and expansion energy in terms of the disintegration map  $f$ .

**Definition 4.6** (Generalized expansion and compression energy). *For any reformation plan  $\gamma = \nu_x \otimes \mu$  of  $\mu$  into  $\nu$  we define the pointwise expansion energy*

$$e_\gamma(x_0) := \limsup_{x \rightarrow x_0} \frac{W(\nu_x, \nu_{x_0})}{|x - x_0|}, \quad (4.4)$$

*and the pointwise compression energy*

$$c_\gamma(x_0) = \limsup_{x \rightarrow x_0} \frac{|x - x_0|}{W(\nu_x, \nu_{x_0})}. \quad (4.5)$$

By using (4.3) we can state

$$e_\gamma(x) = e_f(x), \quad c_\gamma(x) = c_f(x). \quad (4.6)$$

The pointwise reformation energy is then defined by

$$r_\gamma(x_0) = e_\gamma(x_0) + c_\gamma(x_0).$$

*Remark 4.7.* Notice that, since  $W(\delta_x, \delta_y) = |x - y|$ , if  $\gamma$  is a reformation plan induced by a map  $u : X \rightarrow Y$ , say  $\gamma_u = (I \times u)_\# \mu$  and  $f_u$  is the disintegration map of  $\gamma$ , then it results

$$r_\gamma(x_0) = r_{f_u}(x_0) = r_u(x_0).$$

**Definition 4.8.** We define the set  $\text{GRef}(\mu; \nu) \subset \Pi(\mu, \nu)$  as the subset of reformation plans  $\gamma$  of  $\mu$  into  $\nu$  satisfying

$$\forall x_0 \in X : \exists r > 0, H, K \text{ s.t. } e_\gamma(x) \leq K, c_\gamma(x) \leq H \quad (4.7)$$

for every  $x \in X \cap \overline{B}(x_0, r)$ .

*Remark 4.9.* By (4.4)-(4.6) the role played by the disintegration map is clear, hence one is led to argue as in the previous section trying to establish the analogous of Theorem 2.15 in the case of disintegration maps. Unfortunately in the general case of metric spaces some tools as degree theory are not available. Therefore, it is not clear if local invertibility follows by (4.7). Nevertheless, we shall see, analogously to Remark 2.15, that by restricting the analysis to the case of *small reformations*, as defined below, we can prove a global invertibility result suitable to the present case.

**Theorem 4.10.** Let  $\gamma \in \text{GRef}(\mu; \nu)$  and  $f : X \rightarrow \mathcal{P}(Y)$  be the correspondent disintegration map. Assume  $HK < \sqrt[N]{2}$ . Then  $f$  is globally invertible on  $X$ .

*Proof.* Following the proof of Theorem 2.15, by the Metric Area Formula (see [6, 42, 43]) we have

$$\begin{aligned} D\mathcal{H}^N(f(B)) &= \int_{\mathcal{P}(Y)} N(y, f, B) d\mathcal{H}^N(y) = \int_B J(MD(f, x)) dx \leq \\ &\leq \int_B e_f(x)^N dx \leq K^N \mathcal{L}^N(B), \end{aligned}$$

where for any seminorm  $P$  the metric Jacobian is defined by

$$J(P) = N\omega_N \left( \int_{S^{N-1}} P(v)^{-N} d\mathcal{H}^{N-1}(v) \right)^{-1}.$$

Denote by  $g = f|_B$ . Since  $g$  is bi-Lipschitz and  $g(B) \subset f(B)$ , we compute

$$\mathcal{L}^N(B) = \mathcal{L}^N(g^{-1}(g(B))) \leq H^N \mathcal{H}^N(g(B)) \leq H^N \mathcal{H}^N(f(B)).$$

Therefore, we get  $D \leq H^N K^N$ . Then the map  $f$  is globally invertible.  $\square$

**Definition 4.11.** Let us define the set of small reformation plans between  $\mu$  and  $\nu$  as follows

$$\text{GRef}_0(\mu, \nu) = \{\gamma \in \Pi(\mu, \nu) \mid e_\gamma \leq K, c_\gamma \leq H, HK < \sqrt[N]{2}\}. \quad (4.8)$$

## 5. FINDING REFORMATION PLANS

In the following examples we show that it is possible to compare shapes with regular disintegration maps despite no regular transport map does exist.

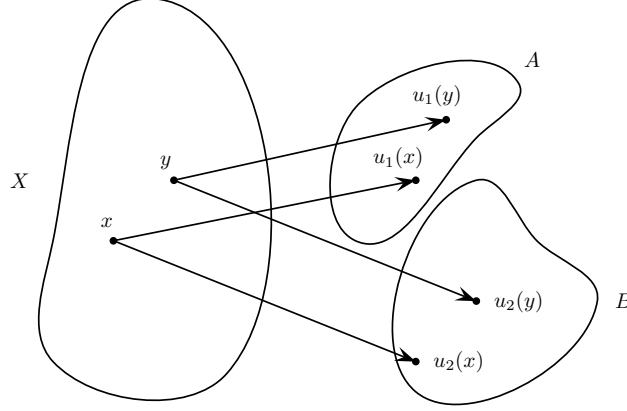


FIGURE 5.1. A disconnected target reformation

*Example 5.1.* Consider a regular domain  $X \subset \mathbb{R}^N$  splitted into  $Y = A \cup B$  for two disjoint regular domains  $A, B \subset \mathbb{R}^N$  in such a way  $1 = \mathcal{L}^N(X) = \mathcal{L}^N(A) + \mathcal{L}^N(B)$ . We find (see [31, 65]) two diffeomorphisms  $u_1 : X \rightarrow A$ ,  $u_2 : X \rightarrow B$  so that  $|Ju_1| = \mathcal{L}^N(A)$ ,  $|Ju_2| = \mathcal{L}^N(B)$ .

Diffeomorphisms with constant Jacobian can be constructed by using the results of [16]. Indeed, let  $\varphi : \Omega \rightarrow \Omega_1$  be a diffeomorphism. Assume for instance  $J\varphi(x) > 0$   $\forall x \in \Omega$  and let  $f(x) = \frac{\mathcal{L}^N(\Omega)}{\mathcal{L}^N(\Omega_1)} J\varphi(x)$ . Then

$$\int_{\Omega} f(x) \, dx = \frac{\mathcal{L}^N(\Omega)}{\mathcal{L}^N(\Omega_1)} \int_{\Omega} J\varphi(x) \, dx = \mathcal{L}^N(\Omega).$$

By the results of [16], there exists a diffeomorphism  $u : \Omega \rightarrow \Omega$  such that  $Ju = f$ . Setting  $\psi = \varphi \circ u^{-1} : \Omega \rightarrow \Omega_1$  it follows that  $J\psi = \frac{\mathcal{L}^N(\Omega)}{\mathcal{L}^N(\Omega_1)}$ .

Let  $\nu_x = \mathcal{L}^N(A)\delta_{u_1(x)} + \mathcal{L}^N(B)\delta_{u_2(x)}$ , then the reformation plan  $\gamma := \nu_x \otimes \mu$  has  $\mu = \mathcal{L}^N \llcorner X$  and  $\nu = \mathcal{L}^N \llcorner Y$  as marginals. We claim that the function  $f(x) = \nu_x$  is, at least locally, bi-Lipschitz. Indeed it results

$$W(\nu_x, \nu_{x_0}) = \mathcal{L}^N(A)|u_1(x) - u_1(x_0)| + \mathcal{L}^N(B)|u_2(x) - u_2(x_0)|.$$

Since  $u_1, u_2$  are diffeomorphisms, we find constants  $K_{1,2}, H_{1,2}, H, K$  such that

$$\begin{aligned} \frac{1}{H}|x - x_0| &\leq \frac{\mathcal{L}^N(A)}{H_1}|x - x_0| + \frac{\mathcal{L}^N(B)}{H_2}|x - x_0| \\ &\leq \mathcal{L}^N(A)|u_1(x) - u_1(x_0)| + \mathcal{L}^N(B)|u_2(x) - u_2(x_0)| \\ &= W(\nu_x, \nu_{x_0}), \end{aligned}$$

$$\begin{aligned}
W(\nu_x, \nu_{x_0}) &= \mathcal{L}^N(A)|u_1(x) - u_1(x_0)| + |\mathcal{L}^N(B)|u_2(x) - u_2(x_0)| \\
&\leq \mathcal{L}^N(A)K_1|x - x_0| + \mathcal{L}^N(B)K_2|x - x_0| \\
&\leq K|x - x_0|.
\end{aligned}$$

*Remark 5.2.* The above construction is possible also for a class of star-shaped domains as in [20, Theorem 5.4] by considering bi-Lipschitz maps in place of diffeomorphisms.

Moreover, generalized reformation maps are useful to compare *near-isometric* shapes.

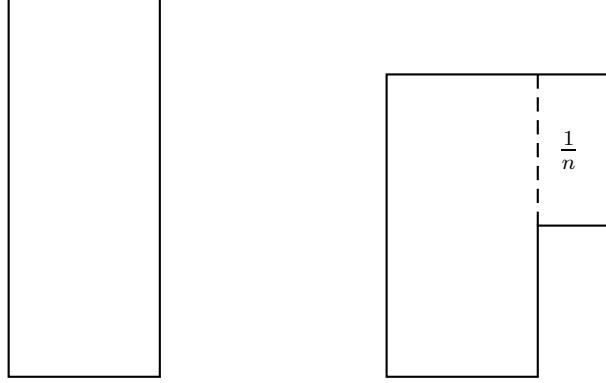


FIGURE 5.2. Bending a rectangle.

*Example 5.3.* Consider a rectangle  $R$  and a bended one with the bended size of  $\frac{1}{n}$ . Consider the maps

$$u_1(x) = \left(1 - \frac{1}{n}\right)(Ax + a), \quad u_2(x) = \frac{1}{n}(Bx + b)$$

for orthogonal matrices  $A, B$  and then the reformation plan

$$\gamma = \left( \left(1 - \frac{1}{n}\right) \delta_{u_1(x)} + \frac{1}{n} \delta_{u_2(x)} \right) \otimes \mu,$$

where  $\mu = \mathcal{L}^N \llcorner R$ . We compute

$$W(\nu_x, \nu_{x_0}) = \left(1 - \frac{1}{n}\right)|u_1(x) - u_1(x_0)| + \frac{1}{n}|u_2(x) - u_2(x_0)| = \left( \left(1 - \frac{1}{n}\right)^2 + \frac{1}{n^2} \right) |x - x_0|.$$

Therefore the function  $f(x) = \nu_x$  is, at least locally, bi-Lipschitz and

$$e_\gamma(x_0) = \left(1 - \frac{1}{n}\right)^2 + \frac{1}{n^2} \rightarrow 1$$

as  $n \rightarrow +\infty$ , while  $c_\gamma(x_0) = \frac{1}{e_\gamma(x_0)}$ .

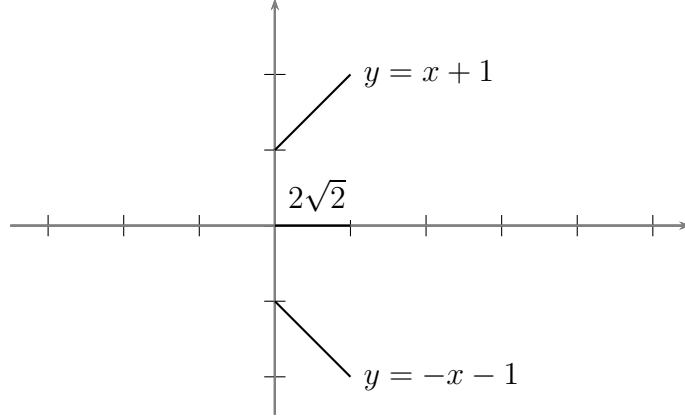


FIGURE 5.3. An horizontal segment, with mass  $2\sqrt{2}$ , splitted into two different ones.

*Example 5.4.* Consider the situation displayed in Figure 5.3. Defining  $\nu_x = \frac{1}{2}(\delta_{x+1} + \delta_{-x-1})$ , we find  $W(\nu_x, \nu_{x_0}) = \sqrt{2}|x - x_0|$ . Hence  $r_\gamma = e_\gamma + c_\gamma = \sqrt{2} + \frac{1}{\sqrt{2}}$ .

*Example 5.5.* Let  $X \subset \mathbb{R}^N$  be a measurable set with  $\mathcal{L}^N(\partial X) = 0$ . We find an increasing sequence of polyhedral sets  $X_n$  such that  $X = \bigcup_{n \geq 1} X_n$  up to a negligible set. Let  $Y \subset \mathbb{R}^N$  be the unitary cube,  $\mathcal{L}^N(X) = \mathcal{L}^N(Y)$  and let  $Y_n \subset Y$  be a rectangle such that  $\mathcal{L}^N(Y_n) = \mathcal{L}^N(X_n) \forall n \in \mathbb{N}$ . Let  $\mu = \mathcal{L}^N \llcorner X$ ,  $\nu = \mathcal{L}^N \llcorner Y$ .

We find a sequence  $(u_n)_{n \in \mathbb{N}}$  so that  $\forall n \in \mathbb{N} \ u_n : X_n \rightarrow Y_n$  is a bi-Lipschitz map with  $Ju_n = 1$ . The volume constraint implies that  $K_n := \text{Lip}(u_n) \leq K$ ,  $H_n := \text{Lip}(u_n^{-1}) \leq H$ . In particular, for every  $x, y \in X_n$  we have

$$\frac{1}{H}|x - y| \leq |u_n(x) - u_n(y)| \leq K|x - y|.$$

By Lipschitz extension, we may consider  $u_n$  as defined on the whole  $X$ . By Ascoli-Arzelá Theorem we find  $u_n \rightarrow u$  uniformly. It follows that

$$\frac{1}{H}|x - y| \leq |u(x) - u(y)| \leq K|x - y|,$$

up to a zero measure set. Moreover

$$\begin{aligned}
 \int_X f(u(x)) \, dx &= \lim_{n \rightarrow +\infty} \int_X f(u_n(x)) \, dx \\
 &= \lim_{n \rightarrow +\infty} \left( \int_{X_n} f(u_n(x)) \, dx + \int_{X \setminus X_n} f(u_n(x)) \, dx \right) \\
 &= \lim_{n \rightarrow +\infty} \int_{Y_n} f(y) \, dy = \int_Y f(y) \, dy.
 \end{aligned}$$

Hence,  $u_{\#}\mu = \nu$ .

## 6. VARIATIONAL PROBLEMS ON GENERALIZED REFORMATIONS

The notion of generalized reformation involves the Lipschitz pointwise constant of maps in a metric space framework. For the associated integral energies it is natural to consider some notion of Sobolev spaces in a metric setting. There exist different notions of such metric Sobolev spaces which coincide provided some mild assumptions such as a *doubling condition*, a Poincarè inequality and a power of integrability  $1 < p < +\infty$  are satisfied. We refer the reader to the Appendix B and the references therein for further informations. In particular the requirement on the power  $1 < p$  will be important to state a general existence result, see Theorem 6.8, for the variational problem related to generalized reformations. Actually, these kind of assumptions seem to form a natural context to work with in the setting of metric analysis. Therefore, along all this section we will assume

$$X = \overline{\Omega} \subset \mathbb{R}^N \text{ compact and satisfying (B.3) and (B.4),} \quad (6.1)$$

$$Y \subset \mathbb{R}^N \text{ compact.}$$

**Definition 6.1.** *Let  $\gamma \in \Pi(\mu, \nu)$ . We define the reformation energy of  $\gamma$  as follows*

$$\mathcal{R}(\gamma) = \int_X (c_\gamma + e_\gamma) \, d\mu. \quad (6.2)$$

*Remark 6.2.* With a slight abuse of notation we are using the same symbol  $\mathcal{R}$  to denote the reformation energy functional defined on the space of reformation maps and the analogous defined on the space of reformation plans. Since in the paper it always appear with its argument specified, there is no risk of confusion.

**Theorem 6.3.** *Let  $\gamma = f(x) \otimes \mu \in \text{GRef}(\mu; \nu)$  be such that  $\mathcal{R}(\gamma) = 2$ ,  $\mu$  absolutely continuous with respect to the Lebesgue measure. Then there exists an open dense subset of  $X$  on which the disintegration map  $f$  is a local isometry (with respect to the Wasserstein distance).*

*Proof.* First observe that since  $X$  is quasiconvex (see for instance [45, Lemma 6.1]), then  $f$  is a Lipschitz function. We have  $e_\gamma = c_\gamma = 1$  a.e. By [51, Prop. 1.1, Sec. 3], there exists an open dense subset  $U \subset X$  on which  $f$  is locally bi-Lipschitz. Therefore,

consider a bi-Lipschitz map  $f : B \rightarrow \mathcal{P}(Y)$  for an open ball  $B \subset U$ . For  $x_1, x_2 \in B$ , by using Fubini Theorem, we find a curve  $\eta$  connecting  $x_1, x_2$  as in [14, Prop. 3.4] in such a way for a.e.  $t$  it results  $e_\gamma(\eta(t)) = 1$  and  $l(\eta) \leq |x_1 - x_2| + \varepsilon$ . Since  $f$  is Lipschitz, the curve  $\rho : [0, 1] \rightarrow (\mathcal{P}(Y), W)$ , defined by  $\rho_t = f(\eta(t))$  is Lipschitz too. Hence, it admits a tangent vector  $v$  (see Theorem A.2). Fixed  $u \in \text{Lip}_1(Y)$ , by standard approximation argument we may suppose that  $u \in \mathcal{C}^1$ . Therefore, by using the continuity equation (A.4) we compute

$$\begin{aligned} \int_Y u \, d(f(x_1) - f(x_2)) &= \int_Y u \, d(\rho_1 - \rho_0) = \int_0^1 \frac{d}{dt} \left( \int_Y u \, d\rho_t \right) dt = \\ &= \int_0^1 \int_Y \langle du, v \rangle d\rho_t \, dt \leq \int_0^1 \int_Y |v| d\rho_t \, dt = \int_0^1 |\dot{\rho}|(t) \, dt \leq \int_0^1 e_\gamma(\eta_t) |\dot{\eta}| dt \leq \\ &\leq l(\eta) \leq |x_1 - x_2| + \varepsilon. \end{aligned}$$

Taking the supremum with respect to  $u$  and letting  $\varepsilon \rightarrow 0^+$  we get the 1-Lipschitz property

$$W(f(x_1), f(x_2)) \leq |x_1 - x_2|.$$

To get the opposite inequality, we argue as follows. Set  $\rho_0 = f(x_1), \rho_1 = f(x_2)$ , let us consider a geodesic  $\rho_t : [0, 1] \rightarrow \mathcal{P}(Y)$  between  $\rho_0$  and  $\rho_1$ , i.e.  $l(\rho) = W(f(x_1), f(x_2))$ . Since  $f$  is bi-Lipschitz, there exists an injective Lipschitz curve  $\gamma : [0, 1] \rightarrow B$  connecting  $x_1, x_2$  such that  $\rho_t = f(\gamma(t))$ . Again by using a Fubini type argument, we find a sequence of Lipschitz injective curves  $(\gamma_n)_{n \in \mathbb{N}}$  so that  $\gamma_n \rightarrow \gamma$  uniformly and  $\text{Lip}(f^{-1})(f(\gamma_n(t))) = 1$  for a.e.  $t \in [0, 1]$ . Therefore, we get  $\sigma_n = f(\gamma_n) \rightarrow \rho$  uniformly in  $(\mathcal{P}(Y), W)$ . By the upper semicontinuity of the Hausdorff measure along the sequence  $\sigma_n$  (see for instance [10, Lemma 4.1]), recalling that for injective curves it results  $l(\sigma) = \mathcal{H}^1(\sigma([0, 1]))$  (see [7]), fixed  $\varepsilon > 0$ , we find a Lipschitz curve  $\sigma$  connecting  $\rho_0$  and  $\rho_1$  such that  $\text{Lip}(f^{-1})(\sigma(t)) = 1$  for a.e.  $t \in [0, 1]$  and  $l(\sigma) \leq W(f(x_1), f(x_2)) + \varepsilon$ . Finally, we compute

$$\begin{aligned} |x_1 - x_2| &= |f^{-1}(\sigma(0)) - f^{-1}(\sigma(1))| = \left| \int_0^1 \frac{d}{dt} f^{-1}(\sigma(t)) \, dt \right| \leq \\ &\leq \int_0^1 |\dot{\sigma}|_W(t) \, dt = l(\sigma) \leq W(f(x_1), f(x_2)) + \varepsilon. \end{aligned}$$

Letting  $\varepsilon \rightarrow 0^+$  we get the thesis.  $\square$

Theorem 6.3 should be compared with Theorem 3.17. The main restriction is on invertibility which is just on an open dense subset. We may say that this open set is of full measure, actually coinciding with the whole space, just for the case of small reformations as done in Theorem 6.4 below. There are different restrictions in doing so for the general case. A first matter relies in characterizing the set where a map is locally invertible on a metric setting. A second one relies on the fact that the integral functional  $\mathcal{R}$  gives a.e. informations, while invertibility requires global conditions.

Therefore the matter is on passing from a.e. conditions to everywhere ones. In the results concerning transport maps, this difficulty was overcome by using degree theory in  $\mathbb{R}^N$ . Therefore, something similar to degree theory over metric spaces should be needed in order to handle with this kind of questions.

Let us introduce the notation

$$\mathcal{E}_G(\mu, \nu) = \inf\{\mathcal{R}(\gamma) \mid \gamma \in \text{GRef}_0(\mu; \nu)\}. \quad (6.3)$$

Concerning symmetry properties of the above generalized reformation energy, the same reasonings made for transport maps, compare with cerning Definition 3.4, hold as well. We remark here that this time the question of symmetry is not just a question on invertibility. For instance, the transport plan  $\gamma = f \otimes \mu$  between  $\mu$  and  $\nu$ , considered in figure 4.1 is isometric, i.e.  $W(f(x), f(x_0)) = |x - x_0|$ . However, reversing the target measures we see that the transport plan between  $\nu$  and  $\mu$  is just locally isometric and no transport plan  $g \otimes \nu$  between  $\nu$  and  $\mu$  is isometric. The fact is that the corresponding disintegration maps are of the form

$$g : Y \rightarrow \mathcal{P}(X).$$

Therefore, symmetry questions are quite involved and here we do not further consider them.

We state the following characterization of the lowest possible value of the generalized reformation energy.

**Theorem 6.4.** *If  $\mathcal{E}_G(\mu, \nu) = 2$ , with  $\mu$  absolutely continuous with respect to the Lebesgue measure, then the infimum is attained at a local isometric reformation plan.*

*Proof.* Let  $\gamma_n$  be a minimizing sequence. By compactness of  $\mathcal{P}(X \times Y)$ , by passing to a subsequence, we may assume that  $\gamma_n \xrightarrow{*} \gamma$ . It follows that  $\gamma$  is also a transport plan between  $\mu$  and  $\nu$ . By disintegration, we also assume that  $\gamma_n = f_n(x) \otimes \mu$ ,  $\gamma = \nu_x \otimes \mu$ . For any fixed  $\varphi \in \mathcal{C}(X)$ ,  $\psi \in \mathcal{C}(Y)$ , we get

$$\begin{aligned} \int_X \varphi(x) \left( \int_Y \psi(y) d\nu_x \right) d\mu &= \int_{X \times Y} \varphi(x) \psi(y) d\gamma = \lim_{n \rightarrow +\infty} \int_{X \times Y} \varphi(x) \psi(y) d\gamma_n = \\ &= \lim_{n \rightarrow +\infty} \int_X \varphi(x) \left( \int_Y \psi(y) df_n(x) \right) d\mu. \end{aligned} \quad (6.4)$$

By density of continuous functions, it follows that  $\int_Y \psi(y) df_n(x) \rightarrow \int_Y \psi(y) d\nu_x$  in Lebesgue spaces of integrable functions.

Since  $X$  is quasiconvex, by definition of generalized reformations, it follows that the sequence  $f_n$  is equi-Lipschitz on  $X$ . By Ascoli-Arzelà Theorem, by passing to a subsequence we have that  $f_n \rightarrow f$  uniformly. Since the disintegration is uniquely determined, it follows that  $f(x) = \nu_x$  for  $\mu$ -a.e.  $x$ . Indeed, since the Wasserstein distance metrizes the weak\* convergence of measures ( $Y$  is compact), for every  $\psi \in$



$\mathcal{C}(Y)$  we have

$$\int_Y \psi df_n(x) \rightarrow \int_Y \psi df(x).$$

Hence, for every  $\varphi \in \mathcal{C}(X)$ , passing to the limit under the integral sign and by (6.4) we get

$$\int_X \varphi(x) \left( \int_Y \psi df(x) \right) d\mu = \lim_{n \rightarrow +\infty} \int_X \varphi(x) \left( \int_Y \psi df_n(x) \right) d\mu = \int_X \varphi(x) \left( \int_Y \psi d\nu_x \right) d\mu.$$

Observe that each  $f_n$  is locally invertible on an open dense  $U_n \subset X$ . Since we are dealing with small reformations, by Theorem 4.10 it follows that each  $f_n$  is globally invertible and locally bi-Lipschitz on  $X$  by Lemma 4.5. By Lemma 3.11, we have that also  $f$  is locally bi-Lipschitz on  $X$ . Moreover, it easily seen that  $f \in \text{GRef}_0(\mu; \nu)$ . Since

$$2 \leq \int_X \left( e_{\gamma_n} + \frac{1}{e_{\gamma_n}} \right) d\mu \leq \mathcal{R}(\gamma_n) \quad \forall n \in \mathbb{N},$$

passing to the limit we get

$$\lim_{n \rightarrow +\infty} \int_X g_n(x) d\mu = 2,$$

where  $g_n(x) = e_{\gamma_n} + \frac{1}{e_{\gamma_n}}$ . Passing to a subsequence we have  $g_n \rightarrow 2$  a.e. Since  $g_n(x) = \varphi(x_n)$  for  $\varphi(t) = t + \frac{1}{t}$ , by continuity of  $\varphi$  it follows that  $e_{\gamma_n} \rightarrow 1$  a.e. On the other hand,  $c_{\gamma_n} \geq \frac{1}{e_{\gamma_n}}$  yielding  $\liminf_{n \rightarrow +\infty} c_{\gamma_n} \geq 1$  a.e. Since  $\gamma_n$  is a minimizing sequence for  $\mathcal{R}$ , we get

$$2 = \lim_{n \rightarrow +\infty} \mathcal{R}(\gamma_n) = 1 + \lim_{n \rightarrow +\infty} \int_X c_{\gamma_n} d\mu.$$

and by Fatou Lemma we infer

$$1 \leq \int_X \liminf_{n \rightarrow +\infty} c_{\gamma_n} d\mu \leq \lim_{n \rightarrow +\infty} \int_X c_{\gamma_n} = 1.$$

Therefore, by passing to a subsequence, we also have that  $c_{\gamma_n} \rightarrow 1$  a.e. Arguing as in the proof of Theorem 6.3, we locally find in  $\Omega$  a curve  $\eta : [0, 1] \rightarrow \mathcal{P}(Y)$  such that  $e_{\gamma_n}(\eta(t)) \rightarrow 1$  a.e. and  $l(\eta) \leq |x_1 - x_2| + \varepsilon$ . Therefore we get

$$W(f_n(x_1), f_n(x_2)) \leq \int_0^1 e_{\gamma_n}(\eta(t)) |\dot{\eta}|(t) dt.$$

Passing to the limit we obtain

$$W(f(x_1), f(x_2)) \leq l(\eta) \leq |x_1 - x_2| + \varepsilon.$$

Letting  $\varepsilon \rightarrow 0^+$  we obtain the 1-Lipschitz condition

$$W(f(x_1), f(x_2)) \leq |x_1 - x_2|.$$

Arguing again as in the proof of Theorem 6.3, we locally obtain

$$W(f(x_1), f(x_2)) = |x_1 - x_2|,$$

hence  $\mathcal{R}(\gamma) = 2$ . □

*Remark 6.5.* To recover a global isometry in the above results as in Theorem 3.17 one should establish some metric version of Liouville Rigidity Theorems as in Theorem 3.8.

A natural question concerns the validity of an existence result as in Theorem 3.12. However, we observe that the approach pursued in the proof of such result involves the push-forward of the transport map. Therefore, for generalized reformations, the push-forward of the disintegrations maps is involved. This point of view leads to consider a variational problem over transport classes as introduced in [30]. The definition of transport classes is the following

**Definition 6.6.** Let  $\gamma, \eta \in \Pi(\mu, \nu)$  with  $\gamma = f(x) \otimes \mu$ ,  $\eta = g(x) \otimes \mu$  be given. We shall say that  $\gamma$  and  $\eta$  are equivalent (by disintegration), in symbols  $\gamma \approx \eta$ , if  $f_{\#}\mu = g_{\#}\mu$ . For any  $\eta \in \Pi(\mu, \nu)$  with  $\eta = g(x) \otimes \mu$ , we shall call transport class any equivalence class of a transport plan  $\eta$  and it will be denoted by  $[\eta]$ , i.e.

$$[\eta] = \{\gamma = f(x) \otimes \mu \mid f_{\#}\mu = g_{\#}\mu\}. \quad (6.5)$$

For a transport map  $u$  the disintegration map is given by  $x \mapsto \delta_{u(x)}$ . In [30] it is shown that every such disintegration map leads to the same push-forwarded measure. In other words, all the reformation plans of the form  $(I \times u)_{\#}\mu$  belong to the same transport class. Moreover, the following result holds true

**Proposition 6.7.** Let  $u : X \rightarrow Y$  be such that  $u_{\#}\mu = \nu$  and let  $\eta = (I \times u)_{\#}\mu = \delta_{u(x)} \otimes \mu$ . If  $\gamma \in [\eta]$  then there exists  $v : X \rightarrow Y$  such that  $\gamma = \delta_{v(x)} \otimes \mu$ , i.e. the transport plan  $\gamma$  is concentrated on the graph of  $v$ .

In this perspective, the variational problem (3.10) studied in section 3 could be rewritten as

$$\text{minimize}\{\mathcal{R}(u) \mid u \in \text{Ref}(\mu; \nu)\} = \text{minimize}_{\text{GRef}(\mu; \nu)}\{\mathcal{R}(\gamma) \mid \gamma \in [\delta_{u(x)} \otimes \mu]\}. \quad (6.6)$$

However, by passing to transport plans, different transport classes arise. By the above discussion it seems natural to fix a transport plan  $\eta \in \Pi(\mu, \nu)$ ,  $\eta = g(x) \otimes \mu$  and to consider the variational problem

$$\text{minimize}_{\text{GRef}(\mu; \nu)}\{\mathcal{R}(\gamma) \mid \gamma \in [\eta]\}. \quad (6.7)$$

**Theorem 6.8.** (Existence of optimal reformation plans) Assume (6.1) and let  $\eta \in \text{GRef}_0(\mu; \nu)$  be given. Then, for every  $p > 1$  the variational problem

$$\text{minimize}_{\text{GRef}_0(\mu; \nu)}\left\{\mathcal{R}^p(\gamma) := \int_X (c_{\gamma}^p + e_{\gamma}^p) d\mu \mid \gamma \in [\eta]\right\} \quad (6.8)$$

admits solutions.

*Proof.* Let  $\gamma_n = f_n(x) \otimes \mu$  be a minimizing sequence. Let  $f_n \rightarrow f$  uniformly with respect to the Wasserstein distance as in the proof of Theorem 6.4. By Lemma 1.4 we get the lower semicontinuity of the term  $\int_X e_\gamma^p(x) d\mu$ . Moreover, by Lemma 3.3 we get

$$\int_X c_\gamma^p(x) d\mu = \int_X \text{Lip}^p(f^{-1})(f(x)) d\mu \quad (6.9)$$

Since (6.1)  $X$  satisfies the doubling condition given in Definition B.4 and the Poincaré inequality given in Definition B.5, we can apply the theory of Sobolev spaces over the subset  $f(X)$  of the metric space  $(\mathcal{P}(Y), W, f_\# \mu)$  (see Appendix B). Moreover (see [56]), since for  $p > 1$  the pointwise Lipschitz constant  $\text{Lip}(g)$  is the minimal generalized upper gradient of the locally Lipschitz map  $g$  ([56, Theorem 5.9]) and the Cheeger  $p$ -energy (B.1) is lower semicontinuous with respect to  $L^p$  convergence ([56, Theorem 2.8]), by using (6.9) we have

$$\int_X c_\gamma^p(x) d\mu = \int_{\mathcal{P}(Y)} \text{Lip}^p(f^{-1})(y) d(f_\# \mu) \leq \liminf_{n \rightarrow +\infty} \int_{\mathcal{P}(Y)} \text{Lip}^p(f_n^{-1})(y) d(f_\# \mu). \quad (6.10)$$

By taking into account the condition  $(f_n)_\# \mu = f_\# \mu \forall n \in \mathbb{N}$ , we get

$$\int_X c_\gamma^p(x) d\mu \leq \liminf_{n \rightarrow +\infty} \int_{\mathcal{P}(Y)} \text{Lip}^p(f_n^{-1})(y) d((f_n)_\# \mu) = \liminf_{n \rightarrow +\infty} \int_X c_{\gamma_n}^p(x) d\mu.$$

□

## APPENDIX A. CURVES IN METRIC SPACES

For reader convenience here we just summarize some basic results. For analysis in metric spaces we refer to [5, 7, 34, 35]. For Lipschitz function on a metric space  $(M, d)$  we introduce the metric derivative according to the following definition.

**Definition A.1.** *Given a curve  $\rho : [a, b] \rightarrow (X, d)$  we define the metric derivative at the point  $t \in ]a, b[$  as the limit*

$$\lim_{h \rightarrow 0} \frac{d(\rho(t+h), \rho(t))}{h} \quad (A.1)$$

*whenever it exists and in this case we denote it by  $|\dot{\rho}|(t)$ .*

Of course, the above notion of metric derivative coincides with the metric differential (1.2). If  $\rho : [a, b] \rightarrow (X, d)$  is a Lipschitz curve, by metric Rademacher Theorem the metric derivative of  $\rho$  exists at  $\mathcal{L}^1$ -a.e. point in  $[a, b]$ . Furthermore, the length of the Lipschitz curve  $\rho$  is given by

$$l(\rho) = \int_a^b |\dot{\rho}|(t) dt. \quad (A.2)$$

We restrict to the case of  $\mathcal{P}(X) := (\mathcal{P}(\Omega), W)$ . The following theorem relates absolutely continuous curves in  $\mathcal{P}(X)$  to the continuity equation.

**Theorem A.2.** *Let  $t \mapsto \rho_t \in \mathcal{P}(X)$ ,  $t \in [0, 1]$ , be a curve. If  $\rho_t$  is absolutely continuous and  $|\dot{\rho}| \in L^1(0, 1)$  is its metric derivative, then there exists a Borel vector field  $v : (t, x) \mapsto v_t(x)$  such that*

$$v_t \in L^p(X, \rho_t) \quad \text{and} \quad \|v_t\|_{L^p(X, \rho_t)} \leq |\dot{\rho}|(t) \quad \text{for } \mathcal{L}^1 - \text{a.e. } t \in [0, 1] \quad (\text{A.3})$$

and the continuity equation

$$\dot{\rho}_t + \operatorname{div}(v \rho_t) = 0 \quad \text{in } (0, 1) \times X, \quad (\text{A.4})$$

where the divergence operator is understood with respect to the spatial variables, is satisfied in the sense of distributions.

Conversely, if  $\rho_t$  satisfies the continuity equation (A.4) for some vector fields  $v_t$  such that  $\|v_t\|_{L^p(\rho_t)} \in L^1(0, 1)$ , then  $t \mapsto \rho_t$  is absolutely continuous and

$$|\dot{\rho}|(t) \leq \|v_t\|_{L^p(X, \rho_t)} \quad \text{for } \mathcal{L}^1 - \text{a.e. } t \in [0, 1].$$

*Remark A.3.* The minimality property (A.3) uniquely determines a tangent field  $v_t$ . We will refer to  $v_t$  as the tangent vector associated to the curve  $t \mapsto \rho_t$ . The continuity equation (A.4) has been used in the Monge-Kantorovich theory since its beginning for many applications. The fact that it characterizes the absolutely continuous curves on the space of probability measures equipped with the Wasserstein metric was only recently pointed out and the full proof is contained in [5].

An immediate consequence of the continuity equation is the following

**Lemma A.4.** *For every solution  $(\rho_t, v_t)$  of the continuity equation (A.4) and for every  $f \in \mathcal{C}^1(X)$  it results*

$$\frac{d}{dt} \left( \int_X f(x) d\rho_t \right) = \int_X \langle d_x f(x), v_t(x) \rangle d\rho_t \quad (\text{A.5})$$

in the sense of distributions.

Actually, it turns out that the map  $f \mapsto \int_X f d\rho_t$  belongs to  $W_{loc}^{1,1}(0, 1)$ . Therefore, formula (A.5) holds for a.e.  $t \in (0, 1)$ . We refer the reader to [4, 5, 29] for proofs and more details.

## APPENDIX B. SOBOLEV SPACES ON METRIC SPACES

There are several ways to generalize the notion of Sobolev spaces into a metric framework, see for instance [12, 18, 33, 36, 45, 56, 61]. The approach based on the notion of *upper gradient* (see [12, 36, 56, 61]) seems to be more appropriate to the context of this paper.

**Definition B.1.** *Let  $(X, d_X)$ ,  $(Y, d_Y)$  be metric spaces, let  $U \subset X$  be an open subset and let  $u : U \rightarrow Y$  be a given map. A Borel function  $g : U \rightarrow [0, +\infty]$  is said to be an upper gradient of  $u$  if for any unit speed curve  $\gamma : [0, l] \rightarrow X$  it results*

$$d_Y(u(\gamma(0)), u(\gamma(l))) \leq \int_0^l g(\gamma(s)) \, ds.$$

If  $u : U \rightarrow Y$  is Lipschitz, then the pointwise Lipschitz constant  $\text{Lip}(u)$  is an upper gradient for  $u$ , see [12, 18, 61]. For  $u \in L^p(U, Y)$ , the *Cheeger type  $p$ -energy* is defined as follows

$$E_p(u) = \inf_{(u_n, g_n)} \liminf_{n \rightarrow +\infty} |g_n|_{L^p}^p, \quad (\text{B.1})$$

where the infimum is taken over the sequences  $(u_n, g_n)$  such that  $g_n$  is an upper gradient of  $u_n$  and  $u_n \rightarrow u, g_n \rightarrow g$  in  $L^p$ . By definition (B.1) it immediately follows

$$E_p(u) \leq \liminf_{n \rightarrow +\infty} E_p(u_n) \quad (\text{B.2})$$

whenever  $u_n \rightarrow u$  in  $L^p$ . The Cheeger metric  $(1, p)$ -Sobolev space is defined as

$$H^{1,p}(U, Y) = \{u \in L^p(U, Y) : E_p(u) < +\infty\}.$$

We need two more definitions.

**Definition B.2.** A function  $g \in L^p$  is called a *generalized upper gradient* for  $u \in H^{1,p}(U, Y)$  if there exists a sequence  $(u_n, g_n)$  such that  $g_n$  is an upper gradient for  $u_n$  and  $u_n \rightarrow u, g_n \rightarrow g$  in  $L^p$ .

From Definition B.1 it follows that  $|g|_{L^p}^p \geq E_p(u)$  whenever  $g$  is a generalized upper gradient for  $u$ .

**Definition B.3.** A generalized upper gradient  $g$  for a map  $u \in H^{1,p}(U, Y)$  is said to be *minimal* if it satisfies  $|g|_{L^p}^p = E_p(u)$

Under some regularity requirement on the target metric space  $Y$ , it may be proved (see [56]) that every  $u \in H^{1,p}(U, Y)$ , with  $1 < p < +\infty$  admits a unique minimal generalized upper gradient  $g_u$ . This minimal generalized upper gradient coincides with the pointwise Lipschitz constant  $\text{Lip}(u)$  under some geometrical property of the spaces  $(X, \mu), Y$  (see [56, Theorem 5.9]). In particular, a crucial role is played by the *doubling condition* and a *weak Poincaré  $(1, p)$ -inequality* for the space  $(X, \mu)$ .

**Definition B.4.** A measure  $\mu$  over a metric space  $X$  is said to be "*doubling*" if  $\mu$  is finite on bounded sets and there exists a constant  $C$  such that for every  $x \in X$  and every  $r > 0$  the following inequality holds

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)). \quad (\text{B.3})$$

**Definition B.5.** Let  $1 \leq p < +\infty$ . A metric measure space  $(X, d, \mu)$  is said to satisfy the *weak Poincaré  $(1, p)$ -inequality* if, for any  $s > 0$ , there exist constants  $C, \Lambda \geq 1$  such that, for any open ball  $B(x, r)$  with  $0 < r \leq s$ , function  $f \in L^1(B(x, \Lambda r))$  and upper gradient  $g : B(x, \Lambda r) \rightarrow [0, +\infty]$  for  $f$ , the following inequality holds

$$\int_{B(x, r)} \left| f - \int_{B(x, r)} f \, d\mu \right| d\mu \leq C \left( \int_{B(x, \Lambda r)} g^p \, d\mu \right)^{\frac{1}{p}} \quad (\text{B.4})$$

Observe that under some geometrical requirement on  $X$ , the Poincaré inequality (B.4) may be required to hold just for Lipschitz functions  $f$  (see [35, 36]). The euclidean space  $\mathbb{R}^N$  equipped with the Lebesgue measure  $\mathcal{L}^N$  is doubling and satisfies the above Poincaré inequality with  $\Lambda = 1$ . Given a square  $Q$  and  $\mu = \mathcal{L}^N \llcorner Q$ , by the inequality

$$\frac{1}{2^N} \mathcal{L}^N(B(x, r)) \leq \mu(B(x, r)) \leq \mathcal{L}^N(B(x, r)),$$

holding for every ball  $B(x, r)$  of  $Q$  and the usual Poincaré inequality on convex sets, it follows that  $(Q, \mu)$  is doubling and supports the Poincaré inequality (B.4). Since the doubling condition and the Poincaré inequality are stable under bi-Lipschitz maps, every diffeomorphic (or bi-Lipschitz), with volume preserving maps, domain  $\Omega$  (as balls, see for instance [20, 31]) with the same volume of the square  $Q$ , equipped with the measure  $\nu = \mathcal{L}^N \llcorner \Omega$  is doubling and supports the Poincaré inequality (B.4). For more details on the doubling and Poincaré inequality we refer the reader for instance to [7, 12, 36, 45].

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